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5 阶变系数 Korteweg – de Vries 方程的光孤子解

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摘　要:KDV 方程可用于描述量子力学、非线性光学、江河等领域中的非均匀传输介质孤立子的传播,是最典型的非线性色散波动方程的代表。以 5 阶变系数 KDV 方程为研究对象,首先结合齐次平衡原理,采用拟设函数法证明了方程当系数满足一定约束条件时存在 sech 函数形式的亮孤子解与 tanh 函数形式的暗孤子解;然后在所得孤子解中结合参数的实际背景,选取了一些特殊参数和方程系数进行了数值模拟,刻画了波函数的实际传播形态。与已有的结果进行比较,发现用此方法更加简洁,研究结果完善了 KDV 方程解的形式,该方法也适用于解决其他非线性波动方程。

关键词:光孤子解;齐次平衡原理;变系数 Korteweg – de Vries 方程;波形图

中图分类号:O175　　**文献标志码**:A

引　言

越来越多有关非线性问题的非线性微分方程已经出现在许多领域,如物理、化学、生物、机械和光学等。例如描述等离子物理中不稳定漂移波的 KdV – Burgers – Kuramoto 方程[1]、描述短脉冲在单模光纤中传播的非线性 Schrödinger 方程[2]、描述单种群优势基因传播的 Fisher 方程[3]等,因此非线性发展方程精确解的研究已经成为一项重要工作。目前,对寻找非线性发展方程的精确解,已经形成较为完善的求解方法,如齐次平衡法[4-5]、首次积分法[6-7]、各种函数展开法和试探函数法[8-9]、逆散射方法[10]、Backlund 变换法[11]等。

KDV 方程可用于描述量子力学、非线性光学、江河等领域中的非均匀传输介质孤立子的传播,以 5 阶时变系数 KDV 方程为研究对象,其形式为:

$$u_t + f(t)u^2u_x + g(t)u_xu_{xx} + h(t)uu_{xxx} + k(t)u_{xxxxx} = 0 \tag{1}$$

其中: $u(x,t)$ 为波函数, $f(t)$, $g(t)$, $h(t)$, $k(t)$ 为实解析函数。文献[12]已经推出该方程的 Lax 对、Darboux 变换和一系列解析解,文献[13]利用 AKNS 变换,构造了方程 $u_t + f(t)u^2u_x + g(t)u_xu_{xx} + h(t)uu_{xxx} + k(t)u_{xxxxx} + l(t)u = 0$ 的自 Backlund 变换和一系列孤子解。

本文利用孤波拟设函数法[14-15]证明当方程(1)的系数满足一定约束条件时,方程存在暗孤子解和亮孤子解,这种方法已经成功运用于若干类非线性偏微分方程[16-18]。

1　变系数 KDV 方程的亮孤子解

假设方程(1)的解具有如下形式:

$$u(x,t) = \lambda(t)\mathrm{sech}^p\xi,\ \xi = l(t)x + v(t)t \tag{2}$$

其中: $\lambda(t)$ 为孤波的振幅, $l(t)$ 为逆宽, $v(t)$ 为波速,这些函数均待定, p 由齐次平衡原理确定。由 $p + 5 = 2p + 3$ 得 $p = 2$。

易得:

$$u_t = \lambda'\mathrm{sech}^2\xi - 2\lambda\mathrm{sech}^2\xi\cdot\tanh\xi\cdot(l'x + v't + v) \tag{3}$$

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$$u_x = -2\lambda l \text{sech}^2\xi \cdot \tanh\xi \tag{4}$$

$$u^2u_x = -2\lambda^3 l \text{sech}^6\xi \cdot \tanh\xi \tag{5}$$

$$u_x u_{xx} = 4\lambda^2 l^3(-2\text{sech}^4\xi \cdot \tanh\xi + 3\text{sech}^6\xi \cdot \tanh\xi) \tag{6}$$

$$u_{xxx} = -2\lambda l^3(4\text{sech}^2\xi \cdot \tanh\xi - 12\text{sech}^4\xi \cdot \tanh\xi) \tag{7}$$

$$u_{xxxxx} = -2\lambda l^5(16\text{sech}^2\xi \cdot \tanh\xi - 240\text{sech}^4\xi \cdot \tanh\xi + 360\text{sech}^6\xi \cdot \tanh\xi) \tag{8}$$

将(3)式~(8)式代入方程(1)得:

$$\lambda'\text{sech}^2\xi - 2\lambda\text{sech}^2\xi \cdot \tanh\xi \cdot (l'x + v't + v)v - 2f(t)\lambda^3 l\text{sech}^6\xi \cdot \tanh\xi + 4g(t)\lambda^2 l^3(-2\text{sech}^4\xi \cdot \tanh\xi + 3\text{sech}^6\xi \cdot \tanh\xi) - 2h(t)\lambda^2 l^3(4\text{sech}^2\xi \cdot \tanh\xi - 12\text{sech}^4\xi \cdot \tanh\xi) - 2k(t)\lambda l^5(16\text{sech}^2\xi \cdot \tanh\xi - 240\text{sech}^4\xi \cdot \tanh\xi + 360\text{sech}^6\xi \cdot \tanh\xi) = 0 \tag{9}$$

令(9)式中 $\text{sech}^i\xi \cdot \tanh^j\xi$ 的系数全为 0 得:

$$\begin{aligned} &\text{sech}^2\xi: \lambda' = 0 \\ &\text{sech}^2\xi \cdot \tanh\xi: l'x + v't + v + 16l^5k(t) = 0 \\ &\text{sech}^4\xi \cdot \tanh\xi: \lambda g(t) + \lambda h(t) - 60l^2k(t) = 0 \\ &\text{sech}^6\xi \cdot \tanh\xi: \lambda^2 f(t) - 6\lambda l^2 g(t) - 12\lambda l^2 h(t) + 360l^4k(t) = 0 \end{aligned} \tag{10}$$

解代数方程组(10)得:

$$\lambda(t) = \lambda_0, l^2(t) = l_0^2 = \frac{\lambda_0 g(t) + \lambda_0 h(t)}{60k(t)}, f(t) = \frac{6l_0^2 h(t)}{\lambda_0}, v(t) = -\frac{16}{t}\int_0^t l^5 k(t)\,\mathrm{d}t$$

从而可知当系数 $f(t), g(t), h(t), k(t)$ 满足 $\frac{f(t)}{h(t)} = \frac{6l_0^2}{\lambda_0^2}$(常数),$\frac{g(t)+h(t)}{k(t)} = \frac{60l_0^2}{\lambda_0}$(常数)时,方程存在亮孤子解:

$$u(x,t) = \lambda_0\text{sech}^2\left(l_0 x - \left(\frac{16}{t}\int_0^t l_0^5 k(t)\,\mathrm{d}t\right)t\right) \tag{11}$$

2 变系数 KDV 方程的暗孤子解

假设方程(1)的解具有如下形式:

$$u(x,t) = \lambda(t)\text{sech}^p\xi, \xi = l(t)x + v(t)t \tag{12}$$

其中,$\lambda(t)$ 为孤波的振幅,$l(t)$ 为逆宽,$v(t)$ 为波速,这些函数均待定。p 由齐次平衡原理确定。由 $p+5 = 2p+3$ 得 $p = 2$。

易得:

$$u_t = \lambda'\tanh^2\xi + 2\lambda(\tanh\xi - \tanh^3\xi) \cdot (l'x + v't + v) \tag{13}$$

$$u_x = 2\lambda l(\tanh\xi - \tanh^3\xi) \tag{14}$$

$$u^2u_x = 2\lambda^3 l(\tanh^5\xi - \tanh^7\xi) \tag{15}$$

$$u_x u_{xx} = 4\lambda^2 l^3(\tanh\xi - 5\tanh^3\xi + 7\tanh^5\xi - 3\tanh^7\xi) \tag{16}$$

$$u_{xxx} = 2\lambda^2 l^3(-8\tanh^3\xi + 20\tanh^5\xi - 12\tanh^7\xi) \tag{17}$$

$$u_{xxxxx} = 2\lambda l^5(136\tanh\xi - 616\tanh^3\xi + 840\tanh^5\xi - 360\tanh^7\xi) \tag{18}$$

将(12)式~(17)式代入方程(1)得:

$$\lambda'\tanh^2\xi + 2\lambda(\tanh\xi - \tanh^3\xi) \cdot (l'x + v't + v) + 2f(t)\lambda^3 l(\tanh^5\xi - \tanh^7\xi) + 4g(t)\lambda^2 l^3(\tanh\xi - 5\tanh^3\xi + 7\tanh^5\xi - 3\tanh^7\xi) + 2h(t)\lambda^2 l^3(-8\tanh^3\xi + 20\tanh^5\xi - 12\tanh^7\xi) + 2k(t)\lambda l^5(136\tanh\xi - 616\tanh^3\xi + 840\tanh^5\xi - 360\tanh^7\xi) = 0 \tag{19}$$

令式(19)中 $\tanh^i\xi$ 的系数全为 0 得:

$$\begin{aligned} &\tanh^2\xi: \lambda' = 0 \\ &\tanh\xi: l'x + v't + v + 2\lambda l^3 g(t) + 136l^5k(t) = 0 \\ &\tanh^3\xi: l'x + v't + v + 10\lambda l^3 g(t) + 8\lambda l^3 h(t) + 616l^5k(t) = 0 \\ &\tanh^5\xi: \lambda^2 f(t) + 14\lambda l^2 g(t) + 20\lambda l^2 h(t) + 840l^4k(t) = 0 \\ &\tanh^7\xi: \lambda^2 f(t) + 6\lambda l^2 g(t) + 12\lambda l^2 h(t) + 360l^4k(t) = 0 \end{aligned} \tag{20}$$

解代数方程组(20)得:

$$\lambda(t) = \lambda_0, l^2(t) = l_0^2 = -\frac{\lambda_0 g(t) + \lambda_0 h(t)}{60k(t)}, f(t) = -\frac{6l_0^2 h(t)}{\lambda_0}, v(t) = -\frac{1}{t}\int_0^t [2\lambda l^3 g(t) + 136l^5k(t)]\,\mathrm{d}t$$

从而可知当系数 $f(t), g(t), h(t), k(t)$ 满足 $\frac{f(t)}{h(t)} = -\frac{6l_0^2}{\lambda_0^2}$(常数),$\frac{g(t)+h(t)}{k(t)} = -\frac{60l_0^2}{\lambda_0}$(常数)时,方程存在暗孤子解:

$$u(x,t) = \lambda_0\tanh^2\left(l_0 x - \left\{\frac{1}{t}\int_0^t [2\lambda l_0^3 g(t) + 136l_0^5 k(t)]\,\mathrm{d}t\right\} t\right) \tag{21}$$

为刻画孤子解对应的波函数实际传播形态,在(11)式和(21)式中结合参数的实际背景,选取了一些特殊参

数和系数进行了数值模拟,图 1、图 2 分别模拟了(11)式和(21)式形式的孤子解的波形图。

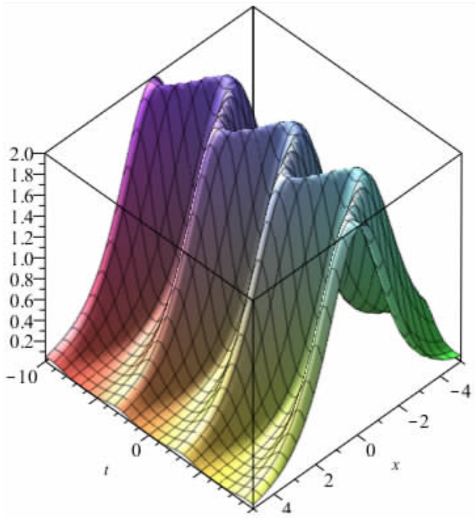

图 1 (11) for λ =2, l=0.5, $k(t)$ =sin$t$

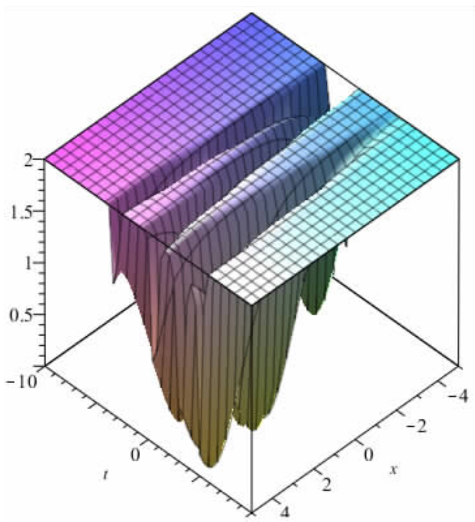

图 2 (21) for λ =2, l=0.5, $k(t)$ =cos$t$

3　结束语

运用拟设函数法获得变系数 5 阶 KDV 方程的光孤子解,包含 sec$h$ 函数形式的亮孤子解和 tan$h$ 函数形式的暗孤子解,扩大了解的范围,对于研究非线性发展方程具有非常广泛的应用意义,这种方法简洁方便,值得推广。

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Optical Soliton Solutions for the Fifth-order Variable-coefficient Korteweg-de Vries Equation

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Abstract: KDV equations can be used to describe the spread of non-uniform transmissive medium soliton in quantum mechanics, nonlinear optics and the field of rivers. It's also the most typical representative of nonlinear dispersive wave equations. In view of the fifth-order variable-coefficient Korteweg-de Vries equation, fist of all, combining with the homogeneous balance principle and using the ansatz method, when the coefficients of equation are constrained by some conditions, the bright solition solutions with sech function and dark solition solutions with tanh function are obtained; Then, combining with the practical background of the parameters in solutions, selected some special parameters and coefficients, the numerical simulation has been conducted as well as the actual communication form of wave function has been depicted. Compared with the past results, the method is more concise, the results generalize and develop the forms of KDV equation's solutions. The method can be aslo used to solve the other non-linear wave equations.

Key words: optical solitons; homogeneous balance method; variable-coefficient Korteweg-de Vries equation