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高斯曲率绝妙定理的几种公式的推导方法

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摘 要:考虑高斯曲率绝妙定理的公式表示问题,运用曲面上基本方程的矩阵表示法,推导出高斯曲率绝妙定理的直接显式公式,指出了高斯曲率隐式公式的验证过程,给出了高斯曲率计算公式Liouville形式的推导过程。

关键词:曲面论基本方程;高斯曲率;高斯绝妙定理

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曲面上高斯曲率的定义和计算公式是经典曲面论的重要内容[1-6]。曲面上的高斯曲率是曲面的内蕴量[1-7],这个著名定理是高斯于 1827 年发现的,称为高斯绝妙定理[2,6]或曲面论的高斯方程,该定理的原始表述形式是用曲面上第一类基本量的隐式表达[1-7]。给出高斯曲率绝妙定理的最终显式表达是研究者的追求,现有文献中给出的推导过程相当繁杂,不利于理解和掌握。研究发现采用曲面论基本方程的矩阵表示法[8-10],运用矩阵运算就可以很简明的推导出高斯曲率绝妙定理的最终显式表达公式[11-12],对高斯曲率计算公式的 Liouville 记忆形式亦给出了推导过程[5]。

1 曲面论基本方程的矩阵方程表形式

给出 C^3 类的正则曲面

$\Sigma:\vec{r} = \vec{r}(u_1, u_2), (u_1, u_2) \in \Delta$

按照文献[1-6,9-10]中的符号体系,给出如下一系列记号,

$$\vec{r}_i = \vec{r}_{u_i} = \frac{\partial \vec{r}}{\partial u_i}$$

$$g_{ij} = \vec{r}_i \cdot \vec{r}_j,\ g_{ij} = g_{ji},\ i,j = 1,2$$

$$g_{11}g_{22} - g_{12}g_{21} = g$$

$$A = (g_{ij}) = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

命

$$A^{-1} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}^{-1} = \frac{1}{g}\begin{pmatrix} g_{22} & -g_{21} \\ -g_{12} & g_{11} \end{pmatrix} = \begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} = (g^{ij})$$

是 $A = (g_{ij})$ 的逆矩阵,

$$\vec{n} = \frac{\vec{r}_1 \times \vec{r}_2}{\|\vec{r}_1 \times \vec{r}_2\|} = \frac{\vec{r}_1 \times \vec{r}_2}{\sqrt{g}}$$

$$\vec{n}_i = \vec{n}_{u_i} = \frac{\partial \vec{n}}{\partial u_i}$$

$$\vec{r}_{ij} = \vec{r}_{u_i u_j} = \frac{\partial^2 \vec{r}}{\partial u_j \partial u_i}$$

$$b_{ij} = \vec{n} \cdot \vec{r}_{ij} = -\vec{n}_j \cdot \vec{r}_i,\ i,j = 1,2,\ b_{ij} = b_{ji}$$

$$B = (b_{ij}) = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

将曲面基本方程改写成矩阵方程形式为[1-6,8-10]:

$$\frac{\partial}{\partial u_1}\begin{pmatrix} \vec{r}_1 \\ \vec{r}_2 \end{pmatrix} = \begin{pmatrix} \Gamma_{11}^1 & \Gamma_{11}^2 \\ \Gamma_{21}^1 & \Gamma_{21}^2 \end{pmatrix}\begin{pmatrix} \vec{r}_1 \\ \vec{r}_2 \end{pmatrix} + \begin{pmatrix} b_{11}\vec{n} \\ b_{21}\vec{n} \end{pmatrix}$$

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$$\frac{\partial}{\partial u_2}\begin{pmatrix}\vec{r}_1\\ \vec{r}_2\end{pmatrix}=\begin{pmatrix}\Gamma_{12}^1 & \Gamma_{12}^2\\ \Gamma_{22}^1 & \Gamma_{22}^2\end{pmatrix}\begin{pmatrix}\vec{r}_1\\ \vec{r}_2\end{pmatrix}+\begin{pmatrix}b_{12}\vec{n}\\ b_{22}\vec{n}\end{pmatrix} \tag{1}$$

$$\begin{pmatrix}\vec{n}_1\\ \vec{n}_2\end{pmatrix}=-\begin{pmatrix}b_1^1 & b_1^2\\ b_2^1 & b_2^2\end{pmatrix}\begin{pmatrix}\vec{r}_1\\ \vec{r}_2\end{pmatrix} \tag{2}$$

其中,

$$\begin{pmatrix}\Gamma_{1j}^1 & \Gamma_{1j}^2\\ \Gamma_{2j}^1 & \Gamma_{2j}^2\end{pmatrix}=\begin{pmatrix}\Gamma_{11j} & \Gamma_{21j}\\ \Gamma_{12j} & \Gamma_{22j}\end{pmatrix}A^{-1}$$

$$\begin{pmatrix}b_1^1 & b_1^2\\ b_2^1 & b_2^2\end{pmatrix}=\begin{pmatrix}b_{11} & b_{21}\\ b_{12} & b_{22}\end{pmatrix}A^{-1}=BA^{-1}$$

$$\Gamma_{lij}=\vec{r}_l\cdot\vec{r}_{ij}=\frac{1}{2}\left(\frac{\partial g_{il}}{\partial u_j}+\frac{\partial g_{jl}}{\partial u_i}-\frac{\partial g_{ij}}{\partial u_l}\right)$$

$$\Gamma_{ij}^k=\sum_{l=1}^{2}g^{kl}\Gamma_{lij},\ \Gamma_{ij}^k=\Gamma_{ji}^k i,j,k=1,2$$

2　曲面论基本方程中系数矩阵满足的方程

对向量 $\vec{r}_i,\vec{n}$ 运用二阶连续偏导数可交换次序的法则,方程组(1)、(2)可解的充要条件是:

$$\frac{\partial^2}{\partial u_2\partial u_1}\begin{pmatrix}\vec{r}_1\\ \vec{r}_2\\ \vec{n}\end{pmatrix}=\frac{\partial^2}{\partial u_1\partial u_2}\begin{pmatrix}\vec{r}_1\\ \vec{r}_2\\ \vec{n}\end{pmatrix}$$

由此,须

$$\frac{\partial}{\partial u_2}\begin{pmatrix}\vec{r}_{11}\\ \vec{r}_{21}\end{pmatrix}=\frac{\partial}{\partial u_1}\begin{pmatrix}\vec{r}_{12}\\ \vec{r}_{22}\end{pmatrix},\frac{\partial}{\partial u_2}\vec{n}_1=\frac{\partial}{\partial u_1}\vec{n}_2$$

利用(1)式,存在可解曲面的充要条件是

$$\frac{\partial}{\partial u_2}\left[\begin{pmatrix}\Gamma_{11}^1 & \Gamma_{11}^2\\ \Gamma_{21}^1 & \Gamma_{21}^2\end{pmatrix}\begin{pmatrix}\vec{r}_1\\ \vec{r}_2\end{pmatrix}+\begin{pmatrix}b_{11}\vec{n}\\ b_{21}\vec{n}\end{pmatrix}\right]=$$

$$\frac{\partial}{\partial u_1}\left[\begin{pmatrix}\Gamma_{12}^1 & \Gamma_{12}^2\\ \Gamma_{22}^1 & \Gamma_{22}^2\end{pmatrix}\begin{pmatrix}\vec{r}_1\\ \vec{r}_2\end{pmatrix}+\begin{pmatrix}b_{12}\vec{n}\\ b_{22}\vec{n}\end{pmatrix}\right] \tag{3}$$

经过代入运算,最后比较左右两端的系数[9-10],可得

$$\frac{\partial}{\partial u_1}\begin{pmatrix}\Gamma_{11}^1 & \Gamma_{11}^2\\ \Gamma_{21}^1 & \Gamma_{21}^2\end{pmatrix}-\frac{\partial}{\partial u_1}\begin{pmatrix}\Gamma_{12}^1 & \Gamma_{12}^2\\ \Gamma_{22}^1 & \Gamma_{22}^2\end{pmatrix}+$$

$$\begin{pmatrix}\Gamma_{11}^1 & \Gamma_{11}^2\\ \Gamma_{21}^1 & \Gamma_{21}^2\end{pmatrix}\begin{pmatrix}\Gamma_{12}^1 & \Gamma_{12}^2\\ \Gamma_{22}^1 & \Gamma_{22}^2\end{pmatrix}-\begin{pmatrix}\Gamma_{12}^1 & \Gamma_{12}^2\\ \Gamma_{22}^1 & \Gamma_{22}^2\end{pmatrix}\begin{pmatrix}\Gamma_{11}^1 & \Gamma_{11}^2\\ \Gamma_{21}^1 & \Gamma_{21}^2\end{pmatrix}=$$

$$\begin{pmatrix}b_{11}\\ b_{21}\end{pmatrix}(b_2^1,b_2^2)-\begin{pmatrix}b_{12}\\ b_{22}\end{pmatrix}(b_1^1,b_1^2) \tag{4}$$

$$\frac{\partial}{\partial u_2}\begin{pmatrix}b_{11}\\ b_{21}\end{pmatrix}-\frac{\partial}{\partial u_1}\begin{pmatrix}b_{12}\\ b_{22}\end{pmatrix}+\begin{pmatrix}\Gamma_{11}^1 & \Gamma_{11}^2\\ \Gamma_{21}^1 & \Gamma_{21}^2\end{pmatrix}\begin{pmatrix}b_{12}\\ b_{22}\end{pmatrix}-$$

$$\begin{pmatrix}\Gamma_{12}^1 & \Gamma_{12}^2\\ \Gamma_{22}^1 & \Gamma_{22}^2\end{pmatrix}\begin{pmatrix}b_{11}\\ b_{21}\end{pmatrix}=0 \tag{5}$$

3　高斯曲率绝妙定理隐式表示公式的矩阵推导方法

对(4)式右端进行代入运算[9-10],可得

$$\frac{\partial}{\partial u_1}\begin{pmatrix}\Gamma_{11}^1 & \Gamma_{11}^2\\ \Gamma_{21}^1 & \Gamma_{21}^2\end{pmatrix}-\frac{\partial}{\partial u_1}\begin{pmatrix}\Gamma_{12}^1 & \Gamma_{12}^2\\ \Gamma_{22}^1 & \Gamma_{22}^2\end{pmatrix}+$$

$$\begin{pmatrix}\Gamma_{11}^1 & \Gamma_{11}^2\\ \Gamma_{21}^1 & \Gamma_{21}^2\end{pmatrix}\begin{pmatrix}\Gamma_{12}^1 & \Gamma_{12}^2\\ \Gamma_{22}^1 & \Gamma_{22}^2\end{pmatrix}-$$

$$\begin{pmatrix}\Gamma_{12}^1 & \Gamma_{12}^2\\ \Gamma_{22}^1 & \Gamma_{22}^2\end{pmatrix}\begin{pmatrix}\Gamma_{11}^1 & \Gamma_{11}^2\\ \Gamma_{21}^1 & \Gamma_{21}^2\end{pmatrix}=$$

$$\begin{pmatrix}0 & b_{11}b_{22}-b_{12}^2\\ b_{12}^2-b_{11}b_{22} & 0\end{pmatrix}A^{-1}=$$

$$\frac{1}{g}\begin{pmatrix}-g_{12}(b_{11}b_{22}-b_{12}^2) & g_{11}(b_{11}b_{22}-b_{12}^2)\\ -g_{22}(b_{11}b_{22}-b_{12}^2) & g_{12}(b_{11}b_{22}-b_{12}^2)\end{pmatrix} \tag{6}$$

由(6)式两边矩阵中右上角的元素对应元素相等,可得[1-6,9-10]

$$\frac{\partial\Gamma_{11}^2}{\partial u_2}-\frac{\partial\Gamma_{12}^2}{\partial u_1}+\Gamma_{11}^1\Gamma_{12}^2+\Gamma_{11}^2\Gamma_{22}^2-\Gamma_{12}^1\Gamma_{11}^2-$$

$$\Gamma_{12}^2\Gamma_{21}^2=(b_{11}b_{22}-b_{12}^2)\frac{g_{11}}{g}$$

于是高斯曲率有内蕴计算公式[1-6,9-10]

$$K=\frac{b_{11}b_{22}-b_{12}^2}{g_{11}g_{22}-g_{12}^2}=\frac{b_{11}b_{22}-b_{12}^2}{g}=$$

$$\frac{1}{g_{11}}\left[\frac{\partial\Gamma_{11}^2}{\partial u_2}-\frac{\partial\Gamma_{12}^2}{\partial u_1}+\Gamma_{11}^1\Gamma_{12}^2+\Gamma_{11}^2\Gamma_{22}^2-\Gamma_{12}^1\Gamma_{11}^2-\Gamma_{12}^2\Gamma_{21}^2\right] \tag{7}$$

由(7)式两边矩阵中左下角的元素对应元素相等,可得

$$\frac{\partial\Gamma_{21}^1}{\partial u_2}-\frac{\partial\Gamma_{22}^1}{\partial u_1}+\Gamma_{21}^1\Gamma_{12}^1+\Gamma_{21}^2\Gamma_{22}^1-\Gamma_{22}^1\Gamma_{11}^1-$$

$$\Gamma_{22}^2\Gamma_{21}^1=-(b_{11}b_{22}-b_{12}^2)\frac{g_{22}}{g}$$

于是高斯曲率内蕴计算公式[1-6,9-10]

$$K=\frac{b_{11}b_{22}-b_{12}^2}{g_{11}g_{22}-g_{12}^2}=\frac{b_{11}b_{22}-b_{12}^2}{g}=-$$

$$\frac{1}{g_{22}}\left[\frac{\partial\Gamma_{21}^2}{\partial u_2}-\frac{\partial\Gamma_{22}^1}{\partial u_1}+\Gamma_{21}^1\Gamma_{12}^1+\Gamma_{21}^2\Gamma_{22}^1-\Gamma_{22}^1\Gamma_{11}^1-\Gamma_{22}^2\Gamma_{21}^1\right] \tag{8}$$

比较(6)式中两边矩阵中的对应元素相等,还可得到另外两个形式的等式[1-4]。

4　高斯曲率绝妙定理的最终显式表示公式的矩阵推导方法

在(6)式左端,利用曲面基本方程中系数矩阵的关

系,经过代入运算[9-10],可得(6)式等价于[9-10]

$$\frac{\partial}{\partial u_2}\begin{pmatrix}\Gamma_{111} & \Gamma_{211}\\ \Gamma_{121} & \Gamma_{221}\end{pmatrix}-\frac{\partial}{\partial u_1}\begin{pmatrix}\Gamma_{112} & \Gamma_{212}\\ \Gamma_{122} & \Gamma_{222}\end{pmatrix}-\begin{pmatrix}\Gamma_{111} & \Gamma_{211}\\ \Gamma_{121} & \Gamma_{221}\end{pmatrix}A^{-1}\begin{pmatrix}\Gamma_{112} & \Gamma_{212}\\ \Gamma_{122} & \Gamma_{222}\end{pmatrix}^T+\begin{pmatrix}\Gamma_{112} & \Gamma_{212}\\ \Gamma_{122} & \Gamma_{222}\end{pmatrix}A^{-1}\begin{pmatrix}\Gamma_{111} & \Gamma_{211}\\ \Gamma_{121} & \Gamma_{221}\end{pmatrix}^T=\begin{pmatrix}0 & b_{11}b_{22}-b_{12}^2\\ b_{12}^2-b_{11}b_{22} & 0\end{pmatrix} \tag{9}$$

由(9)式两边矩阵的右上角对应元素相等,可得

$$\frac{\partial\Gamma_{211}}{\partial u_2}-\frac{\partial\Gamma_{212}}{\partial u_1}-[(\Gamma_{111}g^{11}+\Gamma_{211}g^{21})\Gamma_{122}+(\Gamma_{111}g^{12}+\Gamma_{211}g^{22})\Gamma_{222}]+[(\Gamma_{121}g^{11}+\Gamma_{221}g^{21})\Gamma_{112}+(\Gamma_{121}g^{12}+\Gamma_{221}g^{22})\Gamma_{212}]=b_{11}b_{22}-b_{12}^2 \tag{10}$$

再由

$$K=\frac{b_{11}b_{22}-b_{12}^2}{g_{11}g_{22}-g_{12}^2}=\frac{b_{11}b_{22}-b_{12}^2}{g}$$

$$\begin{pmatrix}g^{11} & g^{12}\\ g^{21} & g^{22}\end{pmatrix}=\frac{1}{g}\begin{pmatrix}g_{22} & -g_{12}\\ -g_{12} & g_{11}\end{pmatrix}$$

可得

$$K=\frac{1}{g^2}\Big\{g\Big(\frac{\partial\Gamma_{211}}{\partial u_2}-\frac{\partial\Gamma_{212}}{\partial u_1}\Big)-[(\Gamma_{111}g_{22}-\Gamma_{211}g_{12})\Gamma_{122}+(-\Gamma_{111}g_{12}+\Gamma_{211}g_{11})\Gamma_{222}]+[(\Gamma_{121}g_{22}-\Gamma_{221}g_{12})\Gamma_{112}+(-\Gamma_{121}g_{12}+\Gamma_{221}g_{11})\Gamma_{212}]\Big\} \tag{11}$$

利用

$$\begin{pmatrix}\Gamma_{111} & \Gamma_{211}\\ \Gamma_{121} & \Gamma_{221}\end{pmatrix}=\begin{pmatrix}\vec r_1\cdot\vec r_{11} & \vec r_2\cdot\vec r_{11}\\ \vec r_1\cdot\vec r_{21} & \vec r_2\cdot\vec r_{21}\end{pmatrix}=\begin{pmatrix}\frac{1}{2}\frac{\partial g_{11}}{\partial u_1} & \frac{\partial g_{12}}{\partial u_1}-\frac{1}{2}\frac{\partial g_{11}}{\partial u_2}\\ \frac{1}{2}\frac{\partial g_{11}}{\partial u_2} & \frac{1}{2}\frac{\partial g_{22}}{\partial u_1}\end{pmatrix}$$

$$\begin{pmatrix}\Gamma_{112} & \Gamma_{212}\\ \Gamma_{122} & \Gamma_{222}\end{pmatrix}=\begin{pmatrix}\vec r_1\cdot\vec r_{12} & \vec r_2\cdot\vec r_{12}\\ \vec r_1\cdot\vec r_{22} & \vec r_2\cdot\vec r_{22}\end{pmatrix}=\begin{pmatrix}\frac{1}{2}\frac{\partial g_{11}}{\partial u_2} & \frac{1}{2}\frac{\partial g_{22}}{\partial u_1}\\ \frac{\partial g_{12}}{\partial u_2}-\frac{1}{2}\frac{\partial g_{22}}{\partial u_1} & \frac{1}{2}\frac{\partial g_{22}}{\partial u_2}\end{pmatrix} \tag{12}$$

从而得出

$$\frac{\partial}{\partial u_2}\begin{pmatrix}\Gamma_{111} & \Gamma_{211}\\ \Gamma_{121} & \Gamma_{221}\end{pmatrix}-\frac{\partial}{\partial u_1}\begin{pmatrix}\Gamma_{112} & \Gamma_{212}\\ \Gamma_{122} & \Gamma_{222}\end{pmatrix}=\begin{pmatrix}0 & \frac{\partial^2 g_{12}}{\partial u_2\partial u_1}-\frac{1}{2}\frac{\partial^2 g_{11}}{\partial u_2\partial u_2}-\frac{1}{2}\frac{\partial^2 g_{22}}{\partial u_1\partial u_1}\\ -\frac{\partial^2 g_{12}}{\partial u_2\partial u_1}-\frac{1}{2}\frac{\partial^2 g_{11}}{\partial u_2\partial u_2}-\frac{1}{2}\frac{\partial^2 g_{22}}{\partial u_1\partial u_1} & 0\end{pmatrix}$$

$$\frac{\partial\Gamma_{211}}{\partial u_2}-\frac{\partial\Gamma_{212}}{\partial u_1}=\frac{\partial^2 g_{12}}{\partial u_2\partial u_1}-\frac{1}{2}\frac{\partial^2 g_{11}}{\partial u_2\partial u_2}-\frac{1}{2}\frac{\partial^2 g_{22}}{\partial u_1\partial u_1} \tag{13}$$

将(12)式、(13)式的对应项代入(11)式,经过计算化简,可得

$$4g^2K=-2g\Big(\frac{\partial^2 g_{11}}{\partial u_2\partial u_2}-2\frac{\partial^2 g_{12}}{\partial u_2\partial u_1}+\frac{\partial^2 g_{22}}{\partial u_1\partial u_1}\Big)+g_{11}\Big(\frac{\partial g_{11}}{\partial u_2}\frac{\partial g_{22}}{\partial u_2}-2\frac{\partial g_{12}}{\partial u_1}\frac{\partial g_{22}}{\partial u_2}+\Big(\frac{\partial g_{22}}{\partial u_1}\Big)^2\Big)+g_{12}\Big(\frac{\partial g_{11}}{\partial u_1}\frac{\partial g_{22}}{\partial u_2}-\frac{\partial g_{11}}{\partial u_2}\frac{\partial g_{22}}{\partial u_1}-2\frac{\partial g_{11}}{\partial u_2}\frac{\partial g_{12}}{\partial u_2}+4\frac{\partial g_{12}}{\partial u_1}\frac{\partial g_{12}}{\partial u_2}-2\frac{\partial g_{12}}{\partial u_1}\frac{\partial g_{22}}{\partial u_1}\Big)+g_{22}\Big(\frac{\partial g_{11}}{\partial u_1}\frac{\partial g_{22}}{\partial u_1}-2\frac{\partial g_{11}}{\partial u_1}\frac{\partial g_{12}}{\partial u_2}+\Big(\frac{\partial g_{11}}{\partial u_2}\Big)^2\Big) \tag{14}$$

公式(14)就是曲面论中著名的高斯方程[11],这里给出了最终的显式公式,可以作为标准形式去验证其他形式。

5 高斯曲率绝妙定理的 Brioschi 公式表示

关于高斯曲率的内蕴量有 Brioschi 公式[1-6,10]:

$$K=\frac{1}{g^2}\left[\begin{vmatrix}g_{11} & g_{12} & \frac{\partial g_{12}}{\partial u_2}-\frac{1}{2}\frac{\partial g_{22}}{\partial u_1}\\ g_{12} & g_{22} & \frac{1}{2}\frac{\partial g_{22}}{\partial u_2}\\ \frac{1}{2}\frac{\partial g_{11}}{\partial u_1} & \frac{\partial g_{12}}{\partial u_1}-\frac{1}{2}\frac{\partial g_{11}}{\partial u_2} & \frac{\partial^2 g_{12}}{\partial u_2\partial u_1}-\frac{1}{2}\frac{\partial^2 g_{11}}{\partial u_2\partial u_2}-\frac{1}{2}\frac{\partial^2 g_{22}}{\partial u_1\partial u_1}\end{vmatrix}-\begin{vmatrix}g_{11} & g_{12} & \frac{1}{2}\frac{\partial g_{11}}{\partial u_2}\\ g_{12} & g_{22} & \frac{1}{2}\frac{\partial g_{22}}{\partial u_1}\\ \frac{1}{2}\frac{\partial g_{11}}{\partial u_2} & \frac{1}{2}\frac{\partial g_{22}}{\partial u_1} & 0\end{vmatrix}\right] \tag{15}$$

将(15)式展开,则得

$$
\begin{aligned}
K = \frac{1}{4g^2}\Big[&-2g\Big(\frac{\partial^2 g_{11}}{\partial u_2\partial u_2} - 2\frac{\partial^2 g_{12}}{\partial u_2\partial u_1} + \frac{\partial^2 g_{22}}{\partial u_1\partial u_1}\Big) + \\
&g_{11}\Big(\frac{\partial g_{11}}{\partial u_2}\frac{\partial g_{22}}{\partial u_2} - 2\frac{\partial g_{12}}{\partial u_1}\frac{\partial g_{22}}{\partial u_2} + \Big(\frac{\partial g_{22}}{\partial u_1}\Big)^2\Big) + \\
&g_{12}\Big(\frac{\partial g_{11}}{\partial u_1}\frac{\partial g_{22}}{\partial u_2} - \frac{\partial g_{11}}{\partial u_2}\frac{\partial g_{22}}{\partial u_1} - 2\frac{\partial g_{11}}{\partial u_2}\frac{\partial g_{12}}{\partial u_2} + \\
&4\frac{\partial g_{12}}{\partial u_1}\frac{\partial g_{12}}{\partial u_2} - 2\frac{\partial g_{12}}{\partial u_1}\frac{\partial g_{22}}{\partial u_1}\Big) + \\
&g_{22}\Big(\frac{\partial g_{11}}{\partial u_1}\frac{\partial g_{22}}{\partial u_1} - 2\frac{\partial g_{11}}{\partial u_1}\frac{\partial g_{12}}{\partial u_2} + \Big(\frac{\partial g_{11}}{\partial u_2}\Big)^2\Big)\Big]
\end{aligned} \tag{16}
$$

显然(14)式与(16)式是一致的。

6 高斯曲率绝妙定理隐式表示公式的验证推导方法

在一般坐标曲线网下,直接验证[1,2,7,9-10],成立

$$
K = \frac{1}{\sqrt{g}}\Big[\frac{\partial}{\partial u_2}\Big(\frac{\sqrt{g}}{g_{11}}\Gamma^2_{11}\Big) - \frac{\partial}{\partial u_1}\Big(\frac{\sqrt{g}}{g_{11}}\Gamma^2_{12}\Big)\Big] \tag{17}
$$

事实上,由

$$
\begin{pmatrix}\Gamma^1_{11} & \Gamma^2_{11}\\ \Gamma^1_{21} & \Gamma^2_{21}\end{pmatrix} = \begin{pmatrix}\Gamma_{111} & \Gamma_{211}\\ \Gamma_{121} & \Gamma_{221}\end{pmatrix}A^{-1} =
$$

$$
\begin{pmatrix}\frac{1}{2}\frac{\partial g_{11}}{\partial u_1} & \frac{\partial g_{12}}{\partial u_1} - \frac{1}{2}\frac{\partial g_{11}}{\partial u_2}\\ \frac{1}{2}\frac{\partial g_{11}}{\partial u_2} & \frac{1}{2}\frac{\partial g_{22}}{\partial u_1}\end{pmatrix}\frac{1}{g}\begin{pmatrix} g_{22} & -g_{12}\\ -g_{12} & g_{11}\end{pmatrix}
$$

得到

$$
\Gamma^2_{11} = \frac{1}{g}\Big[-\frac{1}{2}g_{12}\frac{\partial g_{11}}{\partial u_1} + g_{11}\Big(\frac{\partial g_{12}}{\partial u_1} - \frac{1}{2}\frac{\partial g_{11}}{\partial u_2}\Big)\Big]
$$

$$
\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{g}\Big(-\frac{1}{2}g_{12}\frac{\partial g_{11}}{\partial u_2} + g_{11}\frac{1}{2}\frac{\partial g_{22}}{\partial u_1}\Big) \tag{18}
$$

将(18)式代入(17)式的右端,得到

$$
\begin{aligned}
&\frac{1}{\sqrt{g}}\Big[\frac{\partial}{\partial u_2}\Big(\frac{\sqrt{g}}{g_{11}}\Gamma^2_{11}\Big) - \frac{\partial}{\partial u_1}\Big(\frac{\sqrt{g}}{g_{11}}\Gamma^2_{12}\Big)\Big] = \\
&\frac{1}{\sqrt{g}}\Big[\frac{\partial}{\partial u_2}\Big(\frac{1}{\sqrt{g}g_{11}}\Big(-\frac{1}{2}g_{12}\frac{\partial g_{11}}{\partial u_1} + g_{11}\Big(\frac{\partial g_{12}}{\partial u_1} - \frac{1}{2}\frac{\partial g_{11}}{\partial u_2}\Big)\Big)\Big) - \\
&\frac{\partial}{\partial u_1}\Big(\frac{1}{\sqrt{g}g_{11}}\Big(-\frac{1}{2}g_{12}\frac{\partial g_{11}}{\partial u_2} + g_{11}\frac{1}{2}\frac{\partial g_{22}}{\partial u_1}\Big)\Big)\Big] = \\
&-\frac{1}{2\sqrt{g}}\Big[\frac{\partial}{\partial u_2}\Big(\frac{1}{\sqrt{g}g_{11}}\frac{\partial g_{11}}{\partial u_2}\Big) + \frac{\partial}{\partial u_1}\Big(\frac{1}{\sqrt{g}}\frac{\partial g_{22}}{\partial u_1}\Big) - \\
&2\frac{\partial}{\partial u_2}\Big(\frac{1}{\sqrt{g}}\frac{\partial g_{12}}{\partial u_1}\Big)\Big] - \frac{1}{2\sqrt{g}} \\
&\Big[\frac{\partial}{\partial u_2}\Big(\frac{1}{\sqrt{g}g_{11}}g_{12}\frac{\partial g_{11}}{\partial u_1}\Big) - \frac{\partial}{\partial u_1}\Big(\frac{1}{\sqrt{g}g_{11}}g_{12}\frac{\partial g_{11}}{\partial u_2}\Big)\Big]
\end{aligned} \tag{19}
$$

对(19)式经过求偏导数计算,可以得出(19)式与(16)式相同,于是(17)式成立。

将(17)式写为显式公式,则为

$$
\begin{aligned}
K = &-\frac{1}{2\sqrt{g}}\Big[\frac{\partial}{\partial u_2}\Big(\frac{1}{\sqrt{g}g_{11}}\frac{\partial g_{11}}{\partial u_2}\Big) + \frac{\partial}{\partial u_1}\Big(\frac{1}{\sqrt{g}}\frac{\partial g_{22}}{\partial u_1}\Big) - \\
&2\frac{\partial}{\partial u_2}\Big(\frac{1}{\sqrt{g}}\frac{\partial g_{12}}{\partial u_1}\Big)\Big] - \frac{1}{2\sqrt{g}}\Big[\frac{\partial}{\partial u_2}\Big(\frac{1}{\sqrt{g}g_{11}}g_{12}\frac{\partial g_{11}}{\partial u_1}\Big) - \\
&\frac{\partial}{\partial u_1}\Big(\frac{1}{\sqrt{g}g_{11}}g_{12}\frac{\partial g_{11}}{\partial u_2}\Big)\Big]
\end{aligned} \tag{20}
$$

类似地,在一般坐标曲线网下,直接验证[1,2,7,9-10],成立

$$
K = \frac{1}{\sqrt{g}}\Big[\frac{\partial}{\partial u_1}\Big(\frac{\sqrt{g}}{g_{22}}\Gamma^1_{22}\Big) - \frac{\partial}{\partial u_2}\Big(\frac{\sqrt{g}}{g_{22}}\Gamma^1_{12}\Big)\Big] \tag{21}
$$

事实上,注意到

$$
\Gamma^1_{12} = \frac{1}{g}\Big(\frac{1}{2}g_{22}\frac{\partial g_{11}}{\partial u_2} - g_{12}\frac{1}{2}\frac{\partial g_{22}}{\partial u_1}\Big)
$$

$$
\Gamma^1_{22} = \frac{1}{g}\Big[g_{22}\Big(\frac{\partial g_{12}}{\partial u_2} - \frac{1}{2}\frac{\partial g_{22}}{\partial u_1}\Big) - g_{12}\frac{1}{2}\frac{\partial g_{22}}{\partial u_2}\Big] \tag{22}
$$

将(22)式代入(21)式的右端,得到

$$
\begin{aligned}
&\frac{1}{\sqrt{g}}\Big[\frac{\partial}{\partial u_1}\Big(\frac{\sqrt{g}}{g_{22}}\Gamma^1_{22}\Big) - \frac{\partial}{\partial u_2}\Big(\frac{\sqrt{g}}{g_{22}}\Gamma^1_{12}\Big)\Big] = \\
&\frac{1}{\sqrt{g}}\Big[\frac{\partial}{\partial u_2}\Big(\frac{1}{\sqrt{g}g_{11}}\Big(-\frac{1}{2}g_{12}\frac{\partial g_{11}}{\partial u_1} + g_{11}\Big(\frac{\partial g_{12}}{\partial u_1} - \frac{1}{2}\frac{\partial g_{11}}{\partial u_2}\Big)\Big) - \\
&\frac{\partial}{\partial u_1}\Big(\frac{1}{\sqrt{g}g_{11}}\Big(-\frac{1}{2}g_{12}\frac{\partial g_{11}}{\partial u_2} + g_{11}\frac{1}{2}\frac{\partial g_{22}}{\partial u_1}\Big)\Big)\Big] = \\
&-\frac{1}{2\sqrt{g}}\Big[\frac{\partial}{\partial u_1}\Big(\frac{1}{\sqrt{g}}\frac{\partial g_{22}}{\partial u_1}\Big) + \frac{\partial}{\partial u_2}\Big(\frac{1}{\sqrt{g}}\frac{\partial g_{11}}{\partial u_2}\Big) - \\
&2\frac{\partial}{\partial u_1}\Big(\frac{1}{\sqrt{g}}\frac{\partial g_{12}}{\partial u_2}\Big)\Big] - \\
&\frac{1}{2\sqrt{g}}\Big[\frac{\partial}{\partial u_1}\Big(\frac{1}{\sqrt{g}g_{22}}g_{12}\frac{\partial g_{22}}{\partial u_2}\Big) - \frac{\partial}{\partial u_2}\Big(\frac{1}{\sqrt{g}g_{22}}g_{12}\frac{\partial g_{22}}{\partial u_1}\Big)\Big]
\end{aligned} \tag{23}
$$

对(23)式经过求偏导数计算,可以得到与(16)式同样的式子,从而(21)式成立。于是,有显式公式:

$$
\begin{aligned}
K = &-\frac{1}{2\sqrt{g}}\Big[\frac{\partial}{\partial u_1}\Big(\frac{1}{\sqrt{g}}\frac{\partial g_{22}}{\partial u_1}\Big) + \frac{\partial}{\partial u_2}\Big(\frac{1}{\sqrt{g}}\frac{\partial g_{11}}{\partial u_2}\Big) - \\
&2\frac{\partial}{\partial u_1}\Big(\frac{1}{\sqrt{g}}\frac{\partial g_{12}}{\partial u_2}\Big)\Big] - \\
&\frac{1}{2\sqrt{g}}\Big[\frac{\partial}{\partial u_1}\Big(\frac{1}{\sqrt{g}g_{22}}g_{12}\frac{\partial g_{22}}{\partial u_2}\Big) - \frac{\partial}{\partial u_2}\Big(\frac{1}{\sqrt{g}g_{22}}g_{12}\frac{\partial g_{22}}{\partial u_1}\Big)\Big]
\end{aligned} \tag{24}
$$

7 高斯曲率计算公式的 Liouville 记忆形式的发现推导过程

在曲面 $\Sigma:\vec{r} = \vec{r}(u_1,u_2)$ 上的坐标曲线网是一般网

的情形下,文献[5]中指出了高斯方程有如下的 Liouville 记忆形式:

$$K=-\frac{1}{2\sqrt{g}}\left[\frac{\partial}{\partial u_2}\left(\frac{1}{\sqrt{g}}\frac{\partial g_{11}}{\partial u_2}\right)+\frac{\partial}{\partial u_1}\left(\frac{1}{\sqrt{g}}\frac{\partial g_{22}}{\partial u_1}\right)-\frac{\partial}{\partial u_1}\left(\frac{1}{\sqrt{g}}\frac{\partial g_{12}}{\partial u_2}\right)-\frac{\partial}{\partial u_2}\left(\frac{1}{\sqrt{g}}\frac{\partial g_{12}}{\partial u_1}\right)\right]-\frac{1}{4g^2}\begin{vmatrix} g_{11} & g_{12} & g_{22} \\ \frac{\partial g_{11}}{\partial u_1} & \frac{\partial g_{12}}{\partial u_1} & \frac{\partial g_{22}}{\partial u_1} \\ \frac{\partial g_{11}}{\partial u_2} & \frac{\partial g_{12}}{\partial u_2} & \frac{\partial g_{22}}{\partial u_2} \end{vmatrix} \tag{25}$$

欲证(25)式成立,可以直接验证(25)式的右端与(16)式的形式一样,然而这没有指出(25)式本身是如何发现的。

将(20)式和(24)式的两端相加,得到

$$K=-\frac{1}{2\sqrt{g}}\left[\frac{\partial}{\partial u_2}\left(\frac{1}{\sqrt{g}}\frac{\partial g_{11}}{\partial u_2}\right)+\frac{\partial}{\partial u_1}\left(\frac{1}{\sqrt{g}}\frac{\partial g_{22}}{\partial u_1}\right)-\frac{\partial}{\partial u_1}\left(\frac{1}{\sqrt{g}}\frac{\partial g_{12}}{\partial u_2}\right)-\frac{\partial}{\partial u_2}\left(\frac{1}{\sqrt{g}}\frac{\partial g_{12}}{\partial u_1}\right)\right]-\frac{1}{4\sqrt{g}}\left[\frac{\partial}{\partial u_2}\left(\frac{1}{\sqrt{g}g_{11}}g_{12}\frac{\partial g_{11}}{\partial u_1}\right)+\frac{\partial}{\partial u_1}\left(\frac{1}{\sqrt{g}g_{11}}g_{12}\frac{\partial g_{11}}{\partial u_2}\right)+\frac{\partial}{\partial u_1}\left(\frac{1}{\sqrt{g}g_{22}}g_{12}\frac{\partial g_{22}}{\partial u_2}\right)-\frac{\partial}{\partial u_2}\left(\frac{1}{\sqrt{g}g_{22}}g_{12}\frac{\partial g_{22}}{\partial u_1}\right)\right] \tag{26}$$

经过逐项求偏导数运算及化简,可得到

$$\frac{\partial}{\partial u_2}\left(\frac{1}{\sqrt{g}g_{11}}g_{12}\frac{\partial g_{11}}{\partial u_1}\right)-\frac{\partial}{\partial u_1}\left(\frac{1}{\sqrt{g}g_{11}}g_{12}\frac{\partial g_{11}}{\partial u_2}\right)=\frac{\partial}{\partial u_2}\left(\frac{1}{\sqrt{g}g_{11}}g_{12}\right)\frac{\partial g_{11}}{\partial u_1}-\frac{\partial}{\partial u_1}\left(\frac{1}{\sqrt{g}g_{11}}g_{12}\right)\frac{\partial g_{11}}{\partial u_2}=\frac{1}{2g\sqrt{g}}\left[2g_{22}\left(\frac{\partial g_{11}}{\partial u_1}\frac{\partial g_{12}}{\partial u_2}-\frac{\partial g_{11}}{\partial u_2}\frac{\partial g_{12}}{\partial u_1}\right)-g_{12}\left(\frac{\partial g_{11}}{\partial u_1}\frac{\partial g_{22}}{\partial u_2}-\frac{\partial g_{11}}{\partial u_2}\frac{\partial g_{22}}{\partial u_1}\right)\right] \tag{27}$$

$$\frac{\partial}{\partial u_1}\left(\frac{1}{\sqrt{g}g_{22}}g_{12}\frac{\partial g_{22}}{\partial u_2}\right)-\frac{\partial}{\partial u_2}\left(\frac{1}{\sqrt{g}g_{22}}g_{12}\frac{\partial g_{22}}{\partial u_1}\right)=\frac{\partial}{\partial u_1}\left(\frac{1}{\sqrt{g}g_{22}}g_{12}\right)\frac{\partial g_{22}}{\partial u_2}-\frac{\partial}{\partial u_2}\left(\frac{1}{\sqrt{g}g_{22}}g_{12}\right)\frac{\partial g_{22}}{\partial u_1}=\frac{1}{2g\sqrt{g}}\left[2g_{11}\left(\frac{\partial g_{12}}{\partial u_1}\frac{\partial g_{22}}{\partial u_2}-\frac{\partial g_{12}}{\partial u_2}\frac{\partial g_{22}}{\partial u_1}\right)-g_{12}\left(\frac{\partial g_{11}}{\partial u_1}\frac{\partial g_{22}}{\partial u_2}-\frac{\partial g_{11}}{\partial u_2}\frac{\partial g_{22}}{\partial u_1}\right)\right] \tag{28}$$

将(27)式和(28)式相加,得到

$$\left[\frac{\partial}{\partial u_2}\left(\frac{1}{\sqrt{g}g_{11}}g_{12}\frac{\partial g_{11}}{\partial u_1}\right)-\frac{\partial}{\partial u_1}\left(\frac{1}{\sqrt{g}g_{11}}g_{12}\frac{\partial g_{11}}{\partial u_2}\right)+\frac{\partial}{\partial u_1}\left(\frac{1}{\sqrt{g}g_{22}}g_{12}\frac{\partial g_{22}}{\partial u_2}\right)-\frac{\partial}{\partial u_2}\left(\frac{1}{\sqrt{g}g_{22}}g_{12}\frac{\partial g_{22}}{\partial u_1}\right)\right]=\frac{1}{g\sqrt{g}}\left[g_{11}\left(\frac{\partial g_{12}}{\partial u_1}\frac{\partial g_{22}}{\partial u_2}-\frac{\partial g_{12}}{\partial u_2}\frac{\partial g_{22}}{\partial u_1}\right)-g_{12}\left(\frac{\partial g_{11}}{\partial u_1}\frac{\partial g_{22}}{\partial u_2}-\frac{\partial g_{11}}{\partial u_2}\frac{\partial g_{22}}{\partial u_1}\right)+g_{22}\left(\frac{\partial g_{11}}{\partial u_1}\frac{\partial g_{12}}{\partial u_2}-\frac{\partial g_{11}}{\partial u_2}\frac{\partial g_{12}}{\partial u_1}\right)\right]=\frac{1}{g\sqrt{g}}\begin{vmatrix} g_{11} & g_{12} & g_{22} \\ \frac{\partial g_{11}}{\partial u_1} & \frac{\partial g_{12}}{\partial u_1} & \frac{\partial g_{22}}{\partial u_1} \\ \frac{\partial g_{11}}{\partial u_2} & \frac{\partial g_{12}}{\partial u_2} & \frac{\partial g_{22}}{\partial u_2} \end{vmatrix} \tag{29}$$

将(29)式代入(26)式,得(25)式成立。

8 常用参数坐标网下的高斯曲率绝妙定理的最终表示公式

对 C^3 类正则曲面 $\Sigma:\vec{r}=\vec{r}(u,v),(u,v)\in\Delta$,设曲面的第一基本形式为 $I=E(\mathrm{d}u)^2+2F\mathrm{d}u\mathrm{d}v+G(\mathrm{d}v)^2$,曲面的第二基本形式为 $II=L(\mathrm{d}u)^2+2M\mathrm{d}u\mathrm{d}v+N(\mathrm{d}v)^2$,高斯曲率 $K=\frac{LN-M^2}{EG-F^2}$。

在此参数坐标网的符号体系下,(14)式、(15)式和(25)式的形式分别为:

高斯绝妙定理的最终显式公式:

$$K=\frac{1}{4(EG-F^2)^2}[E(E_vG_v-2F_uG_v+G_u^2)+F(E_uG_v-E_vG_u-2E_vF_v+4F_uF_v-2F_uG_u)+G(E_uG_u-2E_uF_v+E_v^2)-2(EG-F^2)(E_{vv}-2F_{uv}+G_{uu})] \tag{30}$$

高斯绝妙定理的 Brioschi 显式公式:

$$K=\frac{1}{(EG-F^2)^2}\left[\begin{vmatrix} E & F & F_v-\frac{1}{2}G_u \\ F & G & \frac{1}{2}G_v \\ \frac{1}{2}E_u & F_u-\frac{1}{2}E_v & F_{uv}-\frac{1}{2}E_{vv}-\frac{1}{2}G_{uu} \end{vmatrix}-\begin{vmatrix} E & F & \frac{1}{2}E_v \\ F & G & \frac{1}{2}G_u \\ \frac{1}{2}E_v & \frac{1}{2}G_u & 0 \end{vmatrix}\right] \tag{31}$$

将(31)式展开,可得到(30)式。

高斯曲率计算的 Liouville 记忆形式:

$$K = -\frac{1}{2\sqrt{g}}\left[\left(\frac{E_v}{\sqrt{g}}\right)_v + \left(\frac{G_u}{\sqrt{g}}\right)_u - \left(\frac{F_v}{\sqrt{g}}\right)_u - \left(\frac{F_u}{\sqrt{g}}\right)_v\right] - \frac{1}{4g^2}\begin{vmatrix} E & F & G \\ E_u & F_u & G_u \\ E_v & F_v & G_v \end{vmatrix} \tag{32}$$

其中，$g = EG - F^2$。

高斯曲率的内蕴计算公式（30）式，可以作为标准形式检验其他形式，这里给出的证明过程并不复杂，便于查找引用。在文献[12]中，没能正确的写出（30）式，而是写出了一个错误公式，可以用正确的（30）式，给予更正。

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Derivation Methods of Several Formulas of Gaussian Curvature Theorem Egregium

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Abstract: In view of the formula expression problem of Gaussian curvature Theorem Egregium, the direct explicit formula expression of Theorem Egregium of Gaussian curvature is derived by means of the matrix expression of the fundamental equation on curved surface. The proof procedure of Gaussian curvature implicit formula and the derivation of the calculation formula of Gaussian curvature in Liouville form are demonstrated.

Key words: fundamental equation of surface theory; Gaussian curvature; Gauss Theorem Egregium