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变分法的灵敏度分析及其在分布式水文模型的应用

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摘　要:在变分法原理的基础上,处理了变分法的灵敏度分析问题。利用扰动技术推导出边界值问题,运用欧拉－拉格朗日方程、自然边界条件和横截条件,以简单自然的形式推导出变分灵敏度的等价公式。研究了基于变分法的分布式水文模型的灵敏度分析。

关键词:变分原理;欧拉－拉格朗日方程;自然边界条件;横截条件;分布式水文模型

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1　变分问题的解法

变分法是 17 世纪末发展起来的一门数学分支,它理论完整,在力学、物理学、光学、摩擦学、经济学、宇航理论、信息论和自动控制论等诸多方面有广泛的应用。20 世纪中叶发展起来的有限元法,其数学基础之一就是变分法[1]。变分法的关键定理是欧拉－拉格朗日方程,它对应于泛函的临界点[2],变分法最终寻求的是极值函数。变分法在理论物理中非常重要,主要体现在拉格朗日力学中,以及在最小作用原理在量子学的应用中。变分法提供了有限元法的数学基础,它是求解边界值问题的强力工具。纯数学中有在调和函数中使用狄利克雷原理。

考虑如下问题:

$$J(y(t)) = \int_a^b F(t,y(t),y'(t))\mathrm{d}t \tag{1}$$

$$y(t_1) = y_1(t),y(t_2) = y_2(t) \tag{2}$$

假设 $y_0(t)$ 是此问题的一个解,假设 $\delta y(t)$ 满足条件 $\delta y(t_1) = \delta y(t_2) = 0$, 则令 $y(t) = y_0(t) + \varepsilon\delta y(t)$, 且满足条件(2)。将 $y(t)$ 带入问题(1)有

$$J(y(t)) = \int_a^b F(t,y_0(t) + \varepsilon\delta y(t),y'_0(t) + \varepsilon\delta y'(t))\mathrm{d}t \tag{3}$$

这里 ε 为任意实数,则 $J(y(t))$ 就是 ε 的函数,不妨记为 $\prod(\varepsilon)$, (3)式可写为:

$$\prod(\varepsilon) = \int_a^b F(t,y_0(t) + \varepsilon\delta y(t),y'_0(t) + \varepsilon\delta y'(t))\mathrm{d}t$$

当 $\varepsilon = 0$ 时, $\prod(\varepsilon)$ 取得极值,此时$\Pi'(\varepsilon)|_{\varepsilon=0} = 0$。根据复合函数求导法则:

$$\frac{\mathrm{d}\prod(\varepsilon)}{\mathrm{d}\varepsilon} = \int_a^b \Big[\frac{\partial F(t,y_0(t) + \varepsilon\delta y(t),y'_0(t) + \varepsilon\delta y'(t))}{\partial y(t)} \cdot \frac{\mathrm{d}(y_0(t) + \varepsilon\delta y(t))}{\mathrm{d}\varepsilon} + \frac{\partial F(t,y_0(t) + \varepsilon\delta y(t),y'_0(t) + \varepsilon\delta y'(t))}{\partial y'(t)} \cdot \frac{\mathrm{d}(y'_0(t) + \varepsilon\delta y'(t))}{\mathrm{d}\varepsilon}\Big]\mathrm{d}t = \int_a^b \Big[\frac{\partial F(t,y_0(t) + \varepsilon\delta y(t),y'_0(t) + \varepsilon\delta y'(t))}{\partial y(t)} \cdot \delta y(t) + \frac{\partial F(t,y_0(t) + \varepsilon\delta y(t),y'_0(t) + \varepsilon\delta y'(t))}{\partial y'(t)} \cdot \delta y'(t)\Big]\mathrm{d}t$$

令 $\varepsilon = 0$, 有

$$\frac{d\prod(\varepsilon)}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} =$$

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$$\int_a^b \left(\frac{\partial F(t, y_0(t), y'_0(t))}{\partial y(t)} \cdot \delta y(t) + \frac{\partial F(t, y_0(t), y'_0(t))}{\partial y'(t)} \cdot \delta y'(t) \right) \mathrm{d}t$$

于是 $J(y(t))$ 的变分 $\delta J(y(t))$ 为:

$$\delta J(y(t)) = \int_a^b \left(\frac{\partial F(t, y_0(t), y'_0(t))}{\partial y(t)} \cdot \delta y(t) + \frac{\partial F(t, y_0(t), y'_0(t))}{\partial y'(t)} \cdot \delta y'(t) \right) \mathrm{d}t = \int_a^b \left[\frac{\partial F(t, y_0(t), y'_0(t))}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial F(t, y_0(t), y'_0(t))}{\partial y'(t)} \right) \right] \cdot \delta y(t) \mathrm{d}t + \frac{\partial F(t, y_0(t), y'_0(t))}{\partial y'(t)} \delta y(t) \Bigg|_a^b \tag{4}$$

这里使用了分部积分, $\delta y(t)$ 是 $y(t)$ 的偏导数增量。因为 $y_0(t)$ 是使 $J(y(t))$ 取极值的函数,因此

$$\delta J(y(t)) = \int_a^b \left[\frac{\partial F(t, y_0(t), y'_0(t))}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial F(t, y_0(t), y'_0(t))}{\partial y'(t)} \right) \right] \cdot \delta y(t) \mathrm{d}t + \frac{\partial F(t, y_0(t), y'_0(t))}{\partial y'(t)} \delta y(t) \Bigg|_a^b = 0 \tag{5}$$

引理 1 设函数 $y(x)$ 在区间 $[a,b]$ 上连续,任意函数 $\eta(x)$ 在区间 $[a,b]$ 上连续,且当满足条件 $\eta(a) = \eta(b) = 0$ 时,积分 $\int_a^b y(x)\eta(x)dx = 0$ 成立,则在区间 $[a,b]$ 上必有 $y(x) \equiv 0$。

证明 设在 $[a,b]$ 内存在一点 ξ, 使得 $y(x) \neq 0$。由 $y(x)$ 的连续性可知,必存在某个以 ξ 为中心, ρ 为半径的邻域,且在该邻域内 $y(x) \neq 0$。取

$$\eta(x) = \begin{cases} x - \xi & x \in (a,b) \\ 0 & x = a, x = b \end{cases}$$

不难验证 $\eta(x)$ 满足引理 1 的所有条件, 但此时 $\int_a^b y(x)\eta(x)\mathrm{d}x \neq 0$。引理 1 得证。

根据(5)式和引理 1 可以得结论:

(1)对任意的 $\delta y(t)$, 由变分基本原理[3],可得满足极值条件的欧拉 - 拉格朗日方程[4]为:

$$\frac{\partial F(t, y_0(t), y'_0(t))}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial F(t, y_0(t), y'_0(t))}{\partial y'(t)} \right) = 0 \tag{6}$$

(2)设泛函 $J(y(t))$ 的极值曲线 $y = y(t)$ 的一端固定,而另一端在直线 $t = a$ 上移动,则对任意的 $\delta y(a)$,可动的一端必满足自然边界条件[5]:

$$\frac{\partial F(t, y_0(t), y'_0(t))}{\partial y'(t)} \Bigg|_{t=a} = 0 \tag{7}$$

(3)若端点 $t = b$ 固定, $t = a$ 在曲线 $y = \psi(t)$ 上移动,则可动的一端满足横截条件[6]:

$$[F(t, y_0(t), y'_0(t)) + (\psi'(t) - y'_0(t)) F'(t, y_0(t), y'_0(t))] \big|_{t=a} = 0 \tag{8}$$

方程(6) ~ (8)是著名的变分极值[7]的必要条件,在许多实际问题中,都有大量的应用。

2 变分法的灵敏度分析

为了简便化,考虑有限维的情况:

$$J(y(t); \vec{p}) = \int_{a(\vec{p})}^{b(\vec{p})} F(t, y(t), y'(t); \vec{p}) \mathrm{d}t \tag{9}$$

约束条件:

$$H(y(t); \vec{p}) = \vec{0}, \vec{0} \in R^m \tag{10}$$

这里

$$H(y(t); \vec{p}) = (H_1(y(t); \vec{p}), H_2(y(t); \vec{p}), \cdots, H_m(y(t); \vec{p}))\ i = 1, 2, \cdots, m$$

其中

$$H_i(y(t); \vec{p}) = \int_{a(\vec{p})}^{b(\vec{p})} H_i(t, y(t), y'(t); \vec{p}) \mathrm{d}t\ \vec{p} = (p_1, p_2, \cdots, p_k) \in R^k$$

根据约束条件(10)式,以及方程(7)和方程(8),可以得出方程(6)的最优解 $y_0(t)$ 所满足的必要条件。

为得到灵敏度方程,计算与参数有关的目标函数(9)式和(10)式的变分为:

$$\delta J(y_{0(t)}, \vec{p}; \delta y(t), \delta\vec{p}) = \frac{\partial F}{\partial y'(t)}(t, y_0(t), y'_0(t); \vec{p}) \delta y(t) \Bigg|_{a(\vec{p})}^{b(\vec{p})} + F(t, y_0(t), y'_0(t); \vec{p}) \delta(t) \Bigg|_{a(\vec{p})}^{b(\vec{p})} + \int_{a(\vec{p})}^{b(\vec{p})} \varepsilon_y (F(t, y_0(t), y'_0(t); \vec{p})) \delta y(t) \mathrm{d}t + \left(\int_{a(\vec{p})}^{b(\vec{p})} \frac{\partial F}{\partial \vec{p}}(t, y_0(t), y'_0(t); \vec{p}) \right) \cdot \delta\vec{p} \tag{11}$$

$$\delta H_i(y_{0(t)}, \vec{p}; \delta y(t), \delta\vec{p}) = \frac{\partial H_i}{\partial y'(t)}(t, y_0(t), y'_0(t); \vec{p}) \delta y(t) \Bigg|_{a(\vec{p})}^{b(\vec{p})} + H_i(t, y_0(t), y'_0(t); \vec{p}) \delta(t) \Bigg|_{a(\vec{p})}^{b(\vec{p})} + \int_{a(\vec{p})}^{b(\vec{p})} \varepsilon_y (H_i(t, y_0(t), y'_0(t); \vec{p})) \delta y(t) \mathrm{d}t + \left(\int_{a(\vec{p})}^{b(\vec{p})} \frac{\partial H_i}{\partial \vec{p}}(t, y_0(t), y'_0(t); \vec{p}) \right) \cdot \delta\vec{p} = 0 \tag{12}$$

这里

$$\varepsilon_y(F(t,y_0(t),y'_0(t);\vec{p})) = \frac{\partial F}{\partial y(t)}(t,y_0(t),y'_0(t);\vec{p}) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial F}{\partial y'(t)}(t,y_0(t),y'_0(t);\vec{p})\right)$$

引入符号:

$$L(t,y_0(t),y'_0(t),\lambda_0;\vec{p}) = F(t,y_0(t),y'_0(t);\vec{p}) + \lambda_0 \cdot H(t,y_0(t),y'_0(t);\vec{p})$$

则得到以下结论:

(1)拉格朗日方程(6)的变分:

$$\nu(\varepsilon_y(L(t,y_0(t),y'_0(t),\lambda_0;\vec{p}))) = 0 \tag{13}$$

其中符号 ν 定义为:

$$\nu(S) = \frac{\partial S}{\partial y(t)}\delta y(t) + \frac{\partial S}{\partial y'(t)}\delta y'(t) + \frac{\partial S}{\partial y''(t)}\delta y''(t) + \frac{\partial S}{\partial \vec{p}}\delta\vec{p} + \frac{\partial S}{\partial \lambda}\delta\lambda \tag{14}$$

(2)若泛函 $J(y(t))$ 的极值曲线 $y = y(t)$ 一端固定,而另一端在直线 $t = a$ 上移动,则对任意的 $\delta y(a)$,自然边界条件的变分为:

$$\left[\frac{\partial^2 L}{\partial y'(t)\partial y(t)}\delta y(t) + \frac{\partial^2 L}{\partial y''(t)}\delta y'(t) + \frac{\partial^2 L}{\partial y'(t)\partial \vec{p}}\delta\vec{p} + \frac{\partial L}{\partial \lambda}\delta\lambda\right]\Bigg|_{t=a} = 0 \tag{15}$$

(3)若端点 $t = b$ 固定,$t = a$ 在曲线 $\psi(t;\vec{p})$ 上移动,则可动的一端满足横截条件的变分为:

$$\left[\frac{\partial L}{\partial y}\delta y + \frac{\partial L}{\partial y'}\delta y' + \frac{\partial L}{\partial \vec{p}}\delta\vec{p} + \frac{\partial L}{\partial \lambda}\delta\lambda + (\psi' - y')\left(\frac{\partial^2 L}{\partial y'\partial y}\delta y + \frac{\partial^2 L}{\partial y'^2}\delta y' + \frac{\partial^2 L}{\partial y'\partial \vec{p}}\delta\vec{p} + \frac{\partial^2 L}{\partial y'\partial \lambda}\delta\lambda\right) - \frac{\partial L}{\partial y'}(\delta\psi' - \delta y')\right]\Bigg|_{t=a} = 0 \tag{16}$$

$$\delta y(a) = \frac{\partial \psi}{\partial t}(a;\vec{p})\delta(a) + \frac{\partial \psi}{\partial \vec{p}}(a;\vec{p})\delta\vec{p} \tag{17}$$

定理 2[8]　与参数 $\vec{p}$ 有关的原始问题(9)式 ~ (10)式的目标函数的变分灵敏度向量为:

$$\frac{\partial J^*(y;\vec{p})}{\partial \vec{p}} = \left(\int_{a(\vec{p})}^{b(\vec{p})} \frac{\partial L}{\partial \vec{p}}(t,y_0(t),y'_0(t),\lambda_0;\vec{p})\,\mathrm{d}t\right)\cdot \delta\vec{p} + L(t,y_0(t),y'_0(t),\lambda_0;\vec{p})\,|_{b(\vec{p})}\, b'(\vec{p}) - L(t,y_0(t),y'_0(t),\lambda_0;\vec{p})\,|_{a(\vec{p})}\, a'(\vec{p}) + \frac{\partial L}{\partial y'}(t,y_0(t),y'_0(t),\lambda_0;\vec{p})\,|_{b(\vec{p})}\,\delta y'(b(\vec{p}))b'(\vec{p}) - \frac{\partial L}{\partial y'}(t,y_0(t),y'_0(t),\lambda_0;\vec{p})\,|_{a(\vec{p})}\,\delta y'(a(\vec{p}))a'(\vec{p}) \tag{18}$$

定理 2 的直接结果给出了变分灵敏度的一个直接公式,可以直接利用。

3　变分法的灵敏度分析在分布式水文模型的应用

变分法对物理系统的分析和控制提供了一个确定性框架。考虑分布式水文模型[9]的一个简单系统:

$$\text{Minimize } J(x;\alpha) = \int_{t_0}^{t} \varphi(t,x,x';\alpha)\,\mathrm{d}t \tag{19}$$

约束条件: $x(t_0) = 0$。

为分析和解决这个问题,可以直接用变分法。由定理 2 可得

$$\frac{\partial J}{\partial \alpha} = \int_{t_0}^{t} \frac{\partial \varphi(t,x,x';\alpha)}{\partial \alpha}\mathrm{d}t \tag{20}$$

当给出具体的模型与参数时,根据式(20)就可以直接分析研究时间 t 和参数 α 对该系统的影响。

4　结束语

本文从理论上分析了变分法的灵敏度问题,该问题可以用变分引理和泛函极值推导出变分灵敏度的直接公式来解决,此公式可以应用于分布式水文模型及其更多的实际应用中。进一步可以研究当参数变量为无限维时的情况。

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Sensitivity Analysis of Calculus of Variations and Its Application in Distributed Hydrological Models

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Abstract: Based on the principle of variational method, the problem of sensitivity analysis in calculus of variations is managed. A perturbation technique is applied to deduce the boundary value problem, and the Euler-Lagrange equations, natural boundary conditions and transversality conditions are used to get the equivalent formulas for the sensitivities by a simple and natural way. Based on the variational method, the sensitivity analysis of distributed hydrological model is studied.

Key words: variation principle; Euler-Lagrange equations; natural boundary condition; transversality condition; distributed hydrological model

(上接第 96 页)

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Dynamic Interval-Valued Fuzzy Soft Sets and Its Applications in Decision Making

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Abstract: The concept of dynamic interval-valued fuzzy soft sets was proposed. Then, the operation of dynamic interval-valued fuzzy soft sets was defined, and it's natures were studied, next, the decision making method of dynamic interval-valued fuzzy soft sets was proposed. Lastly, an applied example was used to illustrate the feasibility and validity of decision making method.

Key words: soft sets; dynamic interval-valued fuzzy soft sets; dynamic fuzzy soft sets