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限定表面温度的边界层流方程的 Galerkin 有限元数值解

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摘 要:利用一个变换将限定表面温度的边界层流方程转化成二阶边值问题,然后利用 Galerkin 有限元方法将其转化成 n 元非线性方程组,再利用 Newton 迭代法求出在给定初始值和最大误差容忍度的数值解。

关键词:边界层流方程;二阶边值问题;Galerkin 有限元法;Newton 迭代法;数值解

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引 言

自治的三阶非线性微分方程

$$f''' + \frac{m+1}{2}ff'' - mf'^2 = 0, \eta \in [0, +\infty) \quad (1)$$

边界条件

$$f(0) = a, f'(0) = 1, f'(+\infty) = \lim_{\eta \to +\infty} f'(\eta) = 0 \quad (2)$$

是半无限竖直平板上不可压缩流体定常自由对流边界层问题的相似性解[1],其中,$a, m \in R$ 且 m 受限于表面温度。

方程(1)、(2)的解取决于两个参数 a 和 m。当 $m = 0$ 时,方程(1)、(2)就是著名的 Blasius 方程[2];当 $a = 0$ 时,方程(1)、(2)表示流体流经的表面是不可渗透的[3-4];当 $a < 0$ 时,方程(1)、(2)表示流体流经的表面可以注入流体;当 $a > 0$ 时,方程(1)、(2)表示流体流经的表面可以流出流体[4-5]。

关于方程(1)、(2)的解的研究[6-13],可以追溯到一个世纪以前。利用积分运算,Weyl[6] 对方程(1)、(2)进行了严格的分析,但是没有得出解析解。通过引入两种不同的代换,Brighi and Sari[7] 和 Guo and Tsai[8] 将方程(1)、(2)转化成由两个一阶常微分方程组成的自治系统,并且得到解的详细信息。Je - Chiang Tsai[9] 通过分析讨论,得出当 $m \in (-1/3, 0), a \in R$ 时,方程(1)、(2)有唯一有界解;并讨论了当 $m \in (-1/2, -1/3), a \leqslant 0$ 时解的结构。

本文只讨论 $m \in (-1/3, 0), a < 0$ 情形。首先利用一个变换将方程(1)、(2)转化成二阶边值问题,然后利用 Galerkin 有限元方法求出其数值解。

1 二阶边值问题

对于方程(1)、(2),由于 $f'(\eta)$ 在 $[0, +\infty)$ 上单调递增[9],则他必存在单调递增的反函数。于是令:

$$t = f'(\eta), \eta \in [0, +\infty) \quad (3)$$

并记其反函数为 $\eta = g(t), t \in [0,1]$。对(3)式两边关于 t 求导得:

$$1 = f''g'(t), t \in [0,1] \quad (4)$$

记

$$w(t) = f''(\eta), t \in [0,1] \quad (5)$$

对(5)式两边关于 t 求导得:

$$f''' = w'(t)w(t), t \in [0,1] \quad (6)$$

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在 $t = f'(\eta) = f'(g(t))$ 两边同乘 $g'(t)$ 得:

$$\frac{t}{w(t)} = f'(g(t))g'(t), t \in [0,1]$$

对其两边从 t 到 1 积分得:

$$\int_t^1 \frac{s}{w(s)}ds = \int_t^1 f'(g(s))g'(s)ds = f(g(1)) - f(g(t))\ t \in [0,1]$$

由于 $f(g(1)) = f(0) = a$, 则

$$f = a - \int_t^1 \frac{s}{w(s)}ds, t \in [0,1] \tag{7}$$

把式(3)、(5)、(6)和(7)代入方程(1)得:

$$w'(t)w(t) + \frac{m+1}{2}\left(a - \int_t^1 \frac{s}{w(s)}ds\right)w(t) - mt^2 = 0 \quad t \in [0,1]$$

将其两端同除以 $w(t)$ 得:

$$w'(t) = m\frac{t^2}{w(t)} + \frac{m+1}{2}\int_t^1 \frac{s}{w(s)}ds - \frac{m+1}{2}a, t \in [0,1] \tag{8}$$

当 $t = 1$ 时有

$$w'(1) = \frac{m}{w(1)} - \frac{m+1}{2}a \tag{9}$$

对(8)式两端关于 t 求导得

$$w''(t) = \frac{3m-1}{2}\frac{t}{w(t)} - m\frac{t^2 w'(t)}{w^2(t)}, t \in [0,1]$$

将其两端同乘以 $w^2(t)$ 得

$$w^2(t)w''(t) - \frac{3m-1}{2}tw(t) + mt^2 w'(t) = 0, t \in [0,1]$$

又因 $w(0) = f''(+\infty) = 0$, 则将方程(1)、(2)转化成二阶边值问题

$$\begin{cases} w^2(t)w''(t) + mt^2 w'(t) - \frac{3m-1}{2}tw(t) = 0, t \in [0,1] \\ w(0) = 0, w'(1) = \frac{m}{w(1)} - \frac{m+1}{2}a \end{cases} \tag{10}$$

2　有限元方法及其求解方法

2.1　Galerkin 有限元方程组

对于二阶边值问题方程(10),将区间 $[0,1]$ 分成 N 个小区间,步长 $h = \frac{1}{N}$。令 $t_j = jh, j = 0,1,\cdots,N$, 和 $w_j = w(t_j), j = 0,1,\cdots,N$。由 $w(0) = 0$ 知 $w_0 = 0$。令

$$w(t) = \sum_{j=0}^{N} w_j\varphi_j(t), t \in [0,1] \tag{11}$$

其中

$$\varphi_j(t) = \begin{cases} \frac{t - t_{j-1}}{t_j - t_{j-1}}, t \in [t_{j-1}, t_j] \\ \frac{t_{j+1} - t}{t_{j+1} - t_j}, t \in [t_j, t_{j+1}] \\ 0, otherwise \end{cases} \tag{12}$$

由变分原理得方程(10)的 Galerkin 有限元基本公式:

$$\int_0^1 \left[w^2(t)w''(t) + mt^2 w'(t) - \frac{3m-1}{2}tw(t)\right]\varphi_j(t)dt = 0 \quad j = 1,2,\cdots,N \tag{13}$$

由 $w'_N = \frac{m}{w_N} - \frac{m+1}{2}a$ 和 $\varphi_j(0) = \varphi_j(1) = 0 (j = 1,2,\cdots,N-1)$ 得

$$-2\int_0^1 w(w')_j^2 dt - \int_0^1 w^2 w'\varphi'_j dt - \frac{7m-1}{2}\int_0^1 tw_j dt - m\int_0^1 t^2 w'_j dt = 0, j = 1,2,\cdots,N-1 \tag{14}$$

和

$$2mw_N - \frac{m+1}{2}aw_N^2 - 2\int_0^1 w(w')^2\varphi_N dt - \int_0^1 w^2 w'\varphi'_N dt - \frac{7m-1}{2}\int_0^1 tw\varphi_N dt - m\int_0^1 t^2 w\varphi'_N dt = 0 \tag{15}$$

将式(11)、(12)分别代入式(14)、(15)计算得

$$A_j w_{j-1} + B_j w_j + C_j w_{j+1} + \frac{1}{h}(2w_j^3 - w_j^2 w_{j-1} - w_j^2 w_{j+1}) = 0 \quad j = 1,2,\cdots,N-1 \tag{16}$$

其中

$$A_j = \left(\frac{1}{2}t_j^2 - \frac{1}{12}t_j h - \frac{1}{24}h^2\right)m + \frac{h}{24}(2t_j + h)$$

$$B_j = \frac{1}{3}t_j h(5m-1)$$

$$C_j = \left(-\frac{1}{2}t_j^2 - \frac{1}{12}t_j h + \frac{1}{24}h^2\right)m - \frac{h}{24}(2t_j + h)$$

和

$$\left(-\frac{m}{2} + \frac{m+1}{12}h + \frac{m-1}{24}h^2\right)w_{N-1} + \left(-\frac{3m}{2} + \frac{5m-1}{6}h - \frac{5m-1}{24}h^2\right)w_N + \frac{m+1}{2}aw_N^2 + \frac{1}{h}(w_N^3 - w_N^2 w_{N-1}) = 0 \tag{17}$$

2.2　Newton 迭代法求解方程组

令

$$w^T = [w_0, w_1, \cdots, w_N] \tag{18}$$

和

$$H^T(w,m) = [H_1(w,m),H_2(w,m),\cdots,H_N(w,m)] \quad (19)$$

其中

$$H_j(w,m) = A_j w_{j-1} + B_j w_j + C_j w_{j+1} + \frac{1}{h}(2w_j^3 - w_j^2 w_{j-1} - w_j^2 w_{j+1}),j = 1,2,\cdots,N-1$$

和

$$H_N(w,m) = \left(-\frac{m}{2} + \frac{m+1}{12}h + \frac{m-1}{24}h^2\right)w_{N-1} + \left(-\frac{3m}{2} + \frac{5m-1}{6}h - \frac{5m-1}{24}h^2\right)w_N + \frac{m+1}{2}aw_N^2 + \frac{1}{h}(w_N^3 - w_N^2 w_{N-1})$$

求式(16)、(17)的解 w 就是求解如下 $n \times n$ 非线性方程组

$$H(w,m) = 0 \quad (20)$$

的解。记 $H(w,m)$ 的 Jacobian 矩阵为:

$$J_H(w) = \begin{pmatrix} \frac{\partial H_1}{\partial w_1} & \cdots & \frac{\partial H_1}{\partial w_N} \\ \vdots & \ddots & \vdots \\ \frac{\partial H_N}{\partial w_1} & \cdots & \frac{\partial H_N}{\partial w_N} \end{pmatrix}$$

其中

$$\frac{\partial H_j}{\partial w_{j-1}} = A_j - \frac{1}{h}w_j^2$$

$$\frac{\partial H_j}{\partial w_j} = B_j + \frac{2}{h}w_j(3w_j - w_{j-1} - w_{j+1})$$

$$\frac{\partial H_j}{\partial w_{j+1}} = C_j - \frac{1}{h}w_j^2$$

$$\frac{\partial H_j}{\partial w_k} = 0, |j-k| > 1$$

从而矩阵 $J_H(w)$ 是三对角的。

设 w^i 是 w 的第 i 次迭代值的猜测值。假设 $\|H^i\|$ 足够小,可找到一个修正量 Δw^i 满足 $w^{i+1} = w^i + \Delta w^i$,从而使得 $H(w^{i+1}) = 0$。

设 J_H^i 是 w^i 处的雅可比值,则由 Newton 迭代法得 $J_H^i \Delta w^i = -H(w^i,m)$。当 $\|\Delta w^i\|_\infty \le \varepsilon$($\varepsilon$ 是给定的最大误差容忍度)时,上述迭代过程终止。

3 数值结果

选定步长 $h = 0.001$,最大误差容忍度 $\varepsilon = 10^{-8}$,取定初值 $w^0 = [0,0.1,0.1,\cdots,0.1]^T$,再取不同的 a 和 m 计算 $f''(0)(=w(1)=w^N)$ 的值,利用 Matlab 软件编写程序计算数值结果(表 1)。

表 1 $f''(0)$ 的值

am	−0.10	−0.15	−0.18	−0.20	−0.25	−0.30
−0.2	0.00713	0.00862	0.00945	0.00994	0.01112	0.01217
−0.1	0.00704	0.00861	0.00995	0.00988	0.01112	0.01231
0	0.00701	0.00865	0.00952	0.00992	0.01113	0.01231
0.1	0.00729	0.00861	0.00949	0.00995	0.01113	0.01240
0.2	0.00703	0.00823	0.00947	0.00990	0.01128	0.01218

从表 1 可以看出,对任意取定的初值 w^0 和误差容忍度 ε,都可以计算出 $f''(0)$ 的迭代值,并且为正值。

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Galerkin Finite Element Numerical Solutions of Boundary Layer Flows Equation with Prescribed Surface Temperature

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Abstract: By using a transformation, the two order boundary value problem of the boundary layer flows equation with prescribed surface temperature is obtained. And then it is turned into n-dimensional nonlinear equations by utilizing the Galerkin finite element method. After that, the numerical solutions for the nonlinear equations under given value and maximum error tolerance are determined through Newton iterative method.

Key words: boundary layer flows equation; two order boundary value problem; Galerkin finite element method; Newton iterative method; numerical solution