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高斯曲率内蕴公式的几种形式的推导方法

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摘　要:考虑曲面上高斯曲率内蕴公式的表示问题,运用曲面基本方程的矩阵表示法,给出了高斯曲率是内蕴量的直接的显式公式,并指出这种内蕴公式与 Brioschi 的表示公式是明显一致的;给出了高斯曲率简化公式的推导来源,揭示出了高斯曲率隐式公式的发现过程。

关键词:曲面的基本方程;曲面结构方程;高斯曲率;内蕴公式;Brioschi 公式

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曲面上的高斯曲率的定义和计算公式是经典曲面论的重要内容[1-9]。曲面上的高斯曲率是曲面的内蕴量[1-6],这个重要结果是高斯于 1827 年发现的著名定理,称为高斯绝妙定理[2-6],该定理的原始表述形式是用曲面上的第一类基本量的隐式表达。曲面论基本方程的理论到 Riemann 和 Liouville 时代,才被建立了完善体系。意大利数学家 Brioschi 给出了高斯曲率是内蕴量的显式表达公式[1-2,4,6],并给出了正交坐标曲线网下高斯曲率的简化计算公式[1],导致了新型曲面的发现。Brioschi 公式的发现与高斯导出的发现方法完全不同,两者的一致性似乎不能明显的看出来,在曲面论的结构方程的推导过程中还能给出高斯曲率是内蕴量的显式表达公式,这个公式和 Brioschi 公式是明显一致的。高斯曲率是内蕴量的隐式公式[1-7,9],人们通常都是采用验证的方式[7,9],没有指出这种公式是如何合理发现的,本文给出了导致发现的推导过程。对曲面论的基本方程性质的推导,在前人成果的基础上,本文运用矩阵运算的推导方法,给出了简便的推导过程,有利于人们理解掌握,沟通了各部分的联系,构成了一套新的处理体系。

1　曲面论的基本方程的矩阵方程表示形式

曲面论的基本问题是研究由曲面的第一基本形式和第二基本形式如何确定曲面存在的问题,解决的方法是从曲面的基本方程出发,寻找到了存在可解曲面的充要条件[1-6]。给出 C^3 类的正则曲面

$$\Sigma:\vec{r} = \vec{r}(u_1,u_2),(u_1,u_2) \in \Delta$$

按照文献[1-6, 9-10]中的符号体系,给出记号,

$$\vec{r}_i = \vec{r}_{u_i} = \frac{\partial \vec{r}}{\partial u_i}$$

$$g_{ij} = \vec{r}_i \cdot \vec{r}_j,\ g_{ij} = g_{ji},\ i,j = 1,2$$

$$g_{11}g_{22} - g_{12}g_{21} = g$$

$$A = (g_{ij}) = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

命

$$A^{-1} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}^{-1} = \frac{1}{g}\begin{pmatrix} g_{22} & -g_{12} \\ -g_{12} & g_{11} \end{pmatrix} =$$

$$\begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} = (g^{ij})$$

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是 $A = (g_{ij})$ 的逆矩阵,

$$\vec{n} = \frac{\vec{r}_1 \times \vec{r}_2}{\|\vec{r}_1 \times \vec{r}_2\|} = \frac{\vec{r}_1 \times \vec{r}_2}{\sqrt{g}}$$

$$\vec{n}_i = \vec{n}_{u_i} = \frac{\partial \vec{n}}{\partial u_i}$$

$$\vec{r}_{ij} = \vec{r}_{u_i u_j} = \frac{\partial^2 \vec{r}}{\partial u_j \partial u_i}$$

$$b_{ij} = \vec{n} \cdot \vec{r}_{ij} = -\vec{n}_j \cdot \vec{r}_i,\ i,j = 1,2,\ b_{ij} = b_{ji}$$

$$B = (b_{ij}) = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

将曲面的基本方程改写成矩阵方程的形式为[1-6,9]:

$$\frac{\partial}{\partial u_1}\begin{pmatrix} \vec{r}_1 \\ \vec{r}_2 \end{pmatrix} = \begin{pmatrix} \Gamma_{11}^1 & \Gamma_{11}^2 \\ \Gamma_{21}^1 & \Gamma_{21}^2 \end{pmatrix}\begin{pmatrix} \vec{r}_1 \\ \vec{r}_2 \end{pmatrix} + \begin{pmatrix} b_{11}\vec{n} \\ b_{21}\vec{n} \end{pmatrix}$$

$$\frac{\partial}{\partial u_2}\begin{pmatrix} \vec{r}_1 \\ \vec{r}_2 \end{pmatrix} = \begin{pmatrix} \Gamma_{12}^1 & \Gamma_{12}^2 \\ \Gamma_{22}^1 & \Gamma_{22}^2 \end{pmatrix}\begin{pmatrix} \vec{r}_1 \\ \vec{r}_2 \end{pmatrix} + \begin{pmatrix} b_{12}\vec{n} \\ b_{22}\vec{n} \end{pmatrix} \quad (1)$$

$$\begin{pmatrix} \vec{n}_1 \\ \vec{n}_2 \end{pmatrix} = -\begin{pmatrix} b_1^1 & b_1^2 \\ b_2^1 & b_2^2 \end{pmatrix}\begin{pmatrix} \vec{r}_1 \\ \vec{r}_2 \end{pmatrix} \quad (2)$$

其中,

$$\begin{pmatrix} \Gamma_{1j}^1 & \Gamma_{1j}^2 \\ \Gamma_{2j}^1 & \Gamma_{2j}^2 \end{pmatrix} = \begin{pmatrix} \Gamma_{11j} & \Gamma_{21j} \\ \Gamma_{12j} & \Gamma_{22j} \end{pmatrix} A^{-1}$$

$$\begin{pmatrix} b_1^1 & b_1^2 \\ b_2^1 & b_2^2 \end{pmatrix} = \begin{pmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{pmatrix} A^{-1} = BA^{-1}$$

$$\Gamma_{lij} = \vec{r}_l \cdot \vec{r}_{ij} = \frac{1}{2}\left(\frac{\partial g_{il}}{\partial u_j} + \frac{\partial g_{jl}}{\partial u_i} - \frac{\partial g_{ij}}{\partial u_l}\right)$$

2　曲面上第一基本形式矩阵的性质

定理 1[2,6,9]　对曲面 $\vec{r} = \vec{r}(u_1, u_2)$ 上的第一基本形式矩阵 $A = (g_{ij}) = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$,有如下的性质:

(1) $\sum_{\delta=1}^{2} g^{\alpha\delta}\Gamma_{\delta\gamma}^{\beta} + \sum_{\delta=1}^{2} g^{\beta\delta}\Gamma_{\delta\gamma}^{\alpha} = -\frac{\partial g^{\alpha\beta}}{\partial u_\gamma}$

(2) $\frac{\partial g_{\beta\gamma}}{\partial u_\alpha} - \frac{\partial g_{\alpha\gamma}}{\partial u_\beta} = \sum_{\lambda=1}^{2} g_{\beta\lambda}\Gamma_{\alpha\gamma}^{\lambda} - \sum_{\lambda=1}^{2} g_{\alpha\lambda}\Gamma_{\beta\gamma}^{\lambda}$

(3) $\sum_{\beta=1}^{2} \Gamma_{\alpha\beta}^{\beta} = \frac{1}{2}\frac{\partial \ln g}{\partial u_\alpha},\ g = g_{11}g_{22} - (g_{12})^2$

3　曲面基本方程中系数矩阵之间关系的矩阵推导方法

利用曲面的基本方程的矩阵方程表示来研究系数矩阵之间的关系。对向量 $\vec{r}_i, \vec{n}$ 运用二阶连续偏导数可交换次序的法则,方程组(1)、(2)可解的充要条件是

$$\frac{\partial^2}{\partial u_2 \partial u_1}\begin{pmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vec{n} \end{pmatrix} = \frac{\partial^2}{\partial u_1 \partial u_2}\begin{pmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vec{n} \end{pmatrix}$$

由此,须有

$$\frac{\partial}{\partial u_2}\begin{pmatrix} \vec{r}_{11} \\ \vec{r}_{21} \end{pmatrix} = \frac{\partial}{\partial u_1}\begin{pmatrix} \vec{r}_{12} \\ \vec{r}_{22} \end{pmatrix},\ \frac{\partial}{\partial u_2}\vec{n}_1 = \frac{\partial}{\partial u_1}\vec{n}_2$$

利用(1)式,存在可解曲面的充要条件是

$$\frac{\partial}{\partial u_2}\left[\begin{pmatrix} \Gamma_{11}^1 & \Gamma_{11}^2 \\ \Gamma_{21}^1 & \Gamma_{21}^2 \end{pmatrix}\begin{pmatrix} \vec{r}_1 \\ \vec{r}_2 \end{pmatrix} + \begin{pmatrix} b_{11}\vec{n} \\ b_{21}\vec{n} \end{pmatrix}\right] =$$

$$\frac{\partial}{\partial u_1}\left[\begin{pmatrix} \Gamma_{12}^1 & \Gamma_{12}^2 \\ \Gamma_{22}^1 & \Gamma_{22}^2 \end{pmatrix}\begin{pmatrix} \vec{r}_1 \\ \vec{r}_2 \end{pmatrix} + \begin{pmatrix} b_{12}\vec{n} \\ b_{22}\vec{n} \end{pmatrix}\right] \quad (3)$$

(3)式的左端

$$= \frac{\partial}{\partial u_2}\begin{pmatrix} \Gamma_{11}^1 & \Gamma_{11}^2 \\ \Gamma_{21}^1 & \Gamma_{21}^2 \end{pmatrix}\begin{pmatrix} \vec{r}_1 \\ \vec{r}_2 \end{pmatrix} +$$

$$\begin{pmatrix} \Gamma_{11}^1 & \Gamma_{11}^2 \\ \Gamma_{21}^1 & \Gamma_{21}^2 \end{pmatrix}\begin{pmatrix} \vec{r}_{12} \\ \vec{r}_{22} \end{pmatrix} + \begin{pmatrix} \frac{\partial b_{11}}{\partial u_2}\vec{n} \\ \frac{\partial b_{21}}{\partial u_2}\vec{n} \end{pmatrix} + \begin{pmatrix} b_{11}\vec{n}_2 \\ b_{21}\vec{n}_2 \end{pmatrix} =$$

$$\frac{\partial}{\partial u_2}\begin{pmatrix} \Gamma_{11}^1 & \Gamma_{11}^2 \\ \Gamma_{21}^1 & \Gamma_{21}^2 \end{pmatrix}\begin{pmatrix} \vec{r}_1 \\ \vec{r}_2 \end{pmatrix} + \begin{pmatrix} \Gamma_{11}^1 & \Gamma_{11}^2 \\ \Gamma_{21}^1 & \Gamma_{21}^2 \end{pmatrix}\begin{pmatrix} \Gamma_{12}^1 & \Gamma_{12}^2 \\ \Gamma_{22}^1 & \Gamma_{22}^2 \end{pmatrix}\begin{pmatrix} \vec{r}_1 \\ \vec{r}_2 \end{pmatrix} +$$

$$\begin{pmatrix} \Gamma_{11}^1 \\ \Gamma_{21}^1 \end{pmatrix}\begin{pmatrix} b_{12}\vec{n} \\ b_{22}\vec{n} \end{pmatrix} + \begin{pmatrix} \frac{\partial b_{11}}{\partial u_2}\vec{n} \\ \frac{\partial b_{21}}{\partial u_2}\vec{n} \end{pmatrix} - \begin{pmatrix} b_{11}b_2^1 & b_{11}b_2^2 \\ b_{21}b_2^1 & b_{21}b_2^2 \end{pmatrix}\begin{pmatrix} \vec{r}_1 \\ \vec{r}_2 \end{pmatrix}$$

(3)式的右端

$$= \frac{\partial}{\partial u_1}\begin{pmatrix} \Gamma_{12}^1 & \Gamma_{12}^2 \\ \Gamma_{22}^1 & \Gamma_{22}^2 \end{pmatrix}\begin{pmatrix} \vec{r}_1 \\ \vec{r}_2 \end{pmatrix} +$$

$$\begin{pmatrix} \Gamma_{12}^1 & \Gamma_{12}^2 \\ \Gamma_{22}^1 & \Gamma_{22}^2 \end{pmatrix}\begin{pmatrix} \vec{r}_{11} \\ \vec{r}_{21} \end{pmatrix} + \begin{pmatrix} \frac{\partial b_{12}}{\partial u_1}\vec{n} \\ \frac{\partial b_{22}}{\partial u_1}\vec{n} \end{pmatrix} + \begin{pmatrix} b_{12}\vec{n}_1 \\ b_{22}\vec{n}_1 \end{pmatrix} =$$

$$\frac{\partial}{\partial u_1}\begin{pmatrix} \Gamma_{12}^1 & \Gamma_{12}^2 \\ \Gamma_{22}^1 & \Gamma_{22}^2 \end{pmatrix}\begin{pmatrix} \vec{r}_1 \\ \vec{r}_2 \end{pmatrix} + \begin{pmatrix} \Gamma_{12}^1 & \Gamma_{12}^2 \\ \Gamma_{22}^1 & \Gamma_{22}^2 \end{pmatrix}\begin{pmatrix} \Gamma_{11}^1 & \Gamma_{11}^2 \\ \Gamma_{21}^1 & \Gamma_{21}^2 \end{pmatrix}\begin{pmatrix} \vec{r}_1 \\ \vec{r}_2 \end{pmatrix} +$$

$$\begin{pmatrix} \Gamma_{12}^1 & \Gamma_{12}^2 \\ \Gamma_{22}^1 & \Gamma_{22}^2 \end{pmatrix}\begin{pmatrix} b_{11}\vec{n} \\ b_{21}\vec{n} \end{pmatrix} + \begin{pmatrix} \frac{\partial b_{12}}{\partial u_1}\vec{n} \\ \frac{\partial b_{22}}{\partial u_1}\vec{n} \end{pmatrix} - \begin{pmatrix} b_{12}b_1^1 & b_{12}b_1^2 \\ b_{22}b_1^1 & b_{22}b_1^2 \end{pmatrix}\begin{pmatrix} \vec{r}_1 \\ \vec{r}_2 \end{pmatrix}$$

比较(3)式左右两端的系数,可得

$$\frac{\partial}{\partial u_2}\begin{pmatrix}\Gamma_{11}^1 & \Gamma_{11}^2\\ \Gamma_{21}^1 & \Gamma_{21}^2\end{pmatrix}-\frac{\partial}{\partial u_1}\begin{pmatrix}\Gamma_{12}^1 & \Gamma_{12}^2\\ \Gamma_{22}^1 & \Gamma_{22}^2\end{pmatrix}+$$

$$\begin{pmatrix}\Gamma_{11}^1 & \Gamma_{11}^2\\ \Gamma_{21}^1 & \Gamma_{21}^2\end{pmatrix}\begin{pmatrix}\Gamma_{12}^1 & \Gamma_{12}^2\\ \Gamma_{22}^1 & \Gamma_{22}^2\end{pmatrix}-\begin{pmatrix}\Gamma_{12}^1 & \Gamma_{12}^2\\ \Gamma_{22}^1 & \Gamma_{22}^2\end{pmatrix}\begin{pmatrix}\Gamma_{11}^1 & \Gamma_{11}^2\\ \Gamma_{21}^1 & \Gamma_{21}^2\end{pmatrix}=$$

$$\begin{pmatrix}b_{11}\\ b_{21}\end{pmatrix}(b_2^1,\quad b_2^2)-\begin{pmatrix}b_{12}\\ b_{22}\end{pmatrix}(b_1^1,\quad b_1^2) \tag{4}$$

$$\frac{\partial}{\partial u_2}\begin{pmatrix}b_{11}\\ b_{21}\end{pmatrix}-\frac{\partial}{\partial u_1}\begin{pmatrix}b_{12}\\ b_{22}\end{pmatrix}+\begin{pmatrix}\Gamma_{11}^1 & \Gamma_{11}^2\\ \Gamma_{21}^1 & \Gamma_{21}^2\end{pmatrix}\begin{pmatrix}b_{12}\\ b_{22}\end{pmatrix}-$$

$$\begin{pmatrix}\Gamma_{12}^1 & \Gamma_{12}^2\\ \Gamma_{22}^1 & \Gamma_{22}^2\end{pmatrix}\begin{pmatrix}b_{11}\\ b_{21}\end{pmatrix}=0 \tag{5}$$

4 高斯曲率是内蕴量的隐式表示公式的矩阵推导方法

考察(4)式成立的充要条件。(4)式的右端为

$$\begin{pmatrix}b_{11}\\ b_{21}\end{pmatrix}(b_2^1,\quad b_2^2)-\begin{pmatrix}b_{12}\\ b_{22}\end{pmatrix}(b_1^1,\quad b_1^2)=$$

$$\begin{pmatrix}b_{11}\\ b_{21}\end{pmatrix}(b_{21},\quad b_{22})A^{-1}-\begin{pmatrix}b_{12}\\ b_{22}\end{pmatrix}(b_{11},\quad b_{21})A^{-1}=$$

$$\begin{pmatrix}0 & b_{11}b_{22}-b_{12}^2\\ b_{12}^2-b_{11}b_{22} & 0\end{pmatrix}A^{-1} \tag{6}$$

将(6)式代入(4)式,得

$$\frac{\partial}{\partial u_2}\begin{pmatrix}\Gamma_{11}^1 & \Gamma_{11}^2\\ \Gamma_{21}^1 & \Gamma_{21}^2\end{pmatrix}-\frac{\partial}{\partial u_1}\begin{pmatrix}\Gamma_{12}^1 & \Gamma_{12}^2\\ \Gamma_{22}^1 & \Gamma_{22}^2\end{pmatrix}+$$

$$\begin{pmatrix}\Gamma_{11}^1 & \Gamma_{11}^2\\ \Gamma_{21}^1 & \Gamma_{21}^2\end{pmatrix}\begin{pmatrix}\Gamma_{12}^1 & \Gamma_{12}^2\\ \Gamma_{22}^1 & \Gamma_{22}^2\end{pmatrix}-$$

$$\begin{pmatrix}\Gamma_{12}^1 & \Gamma_{12}^2\\ \Gamma_{22}^1 & \Gamma_{22}^2\end{pmatrix}\begin{pmatrix}\Gamma_{11}^1 & \Gamma_{11}^2\\ \Gamma_{21}^1 & \Gamma_{21}^2\end{pmatrix}=$$

$$\begin{pmatrix}0 & b_{11}b_{22}-b_{12}^2\\ b_{12}^2-b_{11}b_{22} & 0\end{pmatrix}A^{-1}=$$

$$\frac{1}{g}\begin{pmatrix}-g_{12}(b_{11}b_{22}-b_{12}^2) & g_{11}(b_{11}b_{22}-b_{12}^2)\\ -g_{22}(b_{11}b_{22}-b_{12}^2) & g_{12}(b_{11}b_{22}-b_{12}^2)\end{pmatrix} \tag{7}$$

由(7)式两边矩阵中右上角的元素对应元素相等,可得[1-6,9]

$$\frac{\partial\Gamma_{11}^2}{\partial u_2}-\frac{\partial\Gamma_{12}^2}{\partial u_1}+\Gamma_{11}^1\Gamma_{12}^2+\Gamma_{11}^2\Gamma_{22}^2-\Gamma_{12}^1\Gamma_{11}^2-$$

$$\Gamma_{12}^2\Gamma_{21}^2=(b_{11}b_{22}-b_{12}^2)\frac{g_{11}}{g}$$

于是高斯曲率的内蕴计算公式[1-6,9]

$$K=\frac{b_{11}b_{22}-b_{12}^2}{g_{11}g_{22}-g_{12}^2}=\frac{b_{11}b_{22}-b_{12}^2}{g}=$$

$$\frac{1}{g_{11}}\Big[\frac{\partial\Gamma_{11}^2}{\partial u_2}-\frac{\partial\Gamma_{12}^2}{\partial u_1}+\Gamma_{11}^1\Gamma_{12}^2+$$

$$\Gamma_{11}^2\Gamma_{22}^2-\Gamma_{12}^1\Gamma_{11}^2-\Gamma_{12}^2\Gamma_{21}^2\Big] \tag{8}$$

由(7)式两边矩阵中左下角的元素对应元素相等,可得

$$\frac{\partial\Gamma_{21}^1}{\partial u_2}-\frac{\partial\Gamma_{22}^1}{\partial u_1}+\Gamma_{21}^1\Gamma_{12}^1+\Gamma_{21}^2\Gamma_{22}^1-\Gamma_{22}^1\Gamma_{11}^1-$$

$$\Gamma_{22}^2\Gamma_{21}^1=-(b_{11}b_{22}-b_{12}^2)\frac{g_{22}}{g}$$

于是高斯曲率有内蕴计算公式[1-6,9]

$$K=\frac{b_{11}b_{22}-b_{12}^2}{g_{11}g_{22}-g_{12}^2}=\frac{b_{11}b_{22}-b_{12}^2}{g}=$$

$$-\frac{1}{g_{22}}\Big[\frac{\partial\Gamma_{21}^1}{\partial u_2}-\frac{\partial\Gamma_{22}^1}{\partial u_1}+\Gamma_{21}^1\Gamma_{12}^1+$$

$$\Gamma_{21}^2\Gamma_{22}^1-\Gamma_{22}^1\Gamma_{11}^1-\Gamma_{22}^2\Gamma_{21}^1\Big] \tag{9}$$

在正交曲线坐标网下,可以求出系数矩阵,然后代入(8)式或(9)式,就可得出高斯曲率的计算公式[2-6]。

比较(7)式中两边矩阵中的对应元素相等,还可得到另外两个形式的等式[1-4]。

5 高斯曲率是内蕴量的显式表示公式的矩阵推导方法

利用曲面基本方程中系数矩阵的关系,(7)式的左端为:

$$\frac{\partial}{\partial u_2}\begin{pmatrix}\Gamma_{11}^1 & \Gamma_{11}^2\\ \Gamma_{21}^1 & \Gamma_{21}^2\end{pmatrix}-\frac{\partial}{\partial u_1}\begin{pmatrix}\Gamma_{12}^1 & \Gamma_{12}^2\\ \Gamma_{22}^1 & \Gamma_{22}^2\end{pmatrix}+$$

$$\begin{pmatrix}\Gamma_{11}^1 & \Gamma_{11}^2\\ \Gamma_{21}^1 & \Gamma_{21}^2\end{pmatrix}\begin{pmatrix}\Gamma_{12}^1 & \Gamma_{12}^2\\ \Gamma_{22}^1 & \Gamma_{22}^2\end{pmatrix}-\begin{pmatrix}\Gamma_{12}^1 & \Gamma_{12}^2\\ \Gamma_{22}^1 & \Gamma_{22}^2\end{pmatrix}\begin{pmatrix}\Gamma_{11}^1 & \Gamma_{11}^2\\ \Gamma_{21}^1 & \Gamma_{21}^2\end{pmatrix}=$$

$$\frac{\partial}{\partial u_2}\begin{pmatrix}\Gamma_{111} & \Gamma_{211}\\ \Gamma_{121} & \Gamma_{221}\end{pmatrix}A^{-1}+\begin{pmatrix}\Gamma_{111} & \Gamma_{211}\\ \Gamma_{121} & \Gamma_{221}\end{pmatrix}\frac{\partial}{\partial u_2}A^{-1}-$$

$$\frac{\partial}{\partial u_1}\begin{pmatrix}\Gamma_{112} & \Gamma_{212}\\ \Gamma_{122} & \Gamma_{222}\end{pmatrix}A^{-1}-\begin{pmatrix}\Gamma_{112} & \Gamma_{212}\\ \Gamma_{122} & \Gamma_{222}\end{pmatrix}\frac{\partial}{\partial u_1}A^{-1}+$$

$$\begin{pmatrix}\Gamma_{111} & \Gamma_{211}\\ \Gamma_{121} & \Gamma_{221}\end{pmatrix}A^{-1}\begin{pmatrix}\Gamma_{112} & \Gamma_{212}\\ \Gamma_{122} & \Gamma_{222}\end{pmatrix}A^{-1}-$$

$$\begin{pmatrix}\Gamma_{112} & \Gamma_{212}\\ \Gamma_{122} & \Gamma_{222}\end{pmatrix}A^{-1}\begin{pmatrix}\Gamma_{111} & \Gamma_{211}\\ \Gamma_{121} & \Gamma_{221}\end{pmatrix}A^{-1} =$$

$$\Big[\frac{\partial}{\partial u_2}\begin{pmatrix}\Gamma_{111} & \Gamma_{211}\\ \Gamma_{121} & \Gamma_{221}\end{pmatrix} - \frac{\partial}{\partial u_1}\begin{pmatrix}\Gamma_{112} & \Gamma_{212}\\ \Gamma_{122} & \Gamma_{222}\end{pmatrix} +$$

$$\begin{pmatrix}\Gamma_{111} & \Gamma_{211}\\ \Gamma_{121} & \Gamma_{221}\end{pmatrix}\frac{\partial A^{-1}}{\partial u_2}A - \begin{pmatrix}\Gamma_{112} & \Gamma_{212}\\ \Gamma_{122} & \Gamma_{222}\end{pmatrix}\frac{\partial A^{-1}}{\partial u_1}A +$$

$$\begin{pmatrix}\Gamma_{111} & \Gamma_{211}\\ \Gamma_{121} & \Gamma_{221}\end{pmatrix}A^{-1}\begin{pmatrix}\Gamma_{112} & \Gamma_{212}\\ \Gamma_{122} & \Gamma_{222}\end{pmatrix} -$$

$$\begin{pmatrix}\Gamma_{112} & \Gamma_{212}\\ \Gamma_{122} & \Gamma_{222}\end{pmatrix}A^{-1}\begin{pmatrix}\Gamma_{111} & \Gamma_{211}\\ \Gamma_{121} & \Gamma_{221}\end{pmatrix}\Big]A^{-1} \qquad (10)$$

利用

$$\frac{\partial A^{-1}}{\partial u_\gamma}A = -A^{-1}\frac{\partial A}{\partial u_\gamma} = -A^{-1}\begin{pmatrix}\Gamma_{11\gamma} & \Gamma_{21\gamma}\\ \Gamma_{12\gamma} & \Gamma_{22\gamma}\end{pmatrix} - A^{-1}\begin{pmatrix}\Gamma_{11\gamma} & \Gamma_{21\gamma}\\ \Gamma_{12\gamma} & \Gamma_{22\gamma}\end{pmatrix}^T$$

于是

$$\begin{pmatrix}\Gamma_{111} & \Gamma_{211}\\ \Gamma_{121} & \Gamma_{221}\end{pmatrix}\frac{\partial A^{-1}}{\partial u_2}A - \begin{pmatrix}\Gamma_{112} & \Gamma_{212}\\ \Gamma_{122} & \Gamma_{222}\end{pmatrix}\frac{\partial A^{-1}}{\partial u_1}A = -$$

$$\begin{pmatrix}\Gamma_{111} & \Gamma_{211}\\ \Gamma_{121} & \Gamma_{221}\end{pmatrix}A^{-1}\begin{pmatrix}\Gamma_{112} & \Gamma_{212}\\ \Gamma_{122} & \Gamma_{222}\end{pmatrix} -$$

$$\begin{pmatrix}\Gamma_{111} & \Gamma_{211}\\ \Gamma_{121} & \Gamma_{221}\end{pmatrix}A^{-1}\begin{pmatrix}\Gamma_{112} & \Gamma_{212}\\ \Gamma_{122} & \Gamma_{222}\end{pmatrix}^T +$$

$$\begin{pmatrix}\Gamma_{112} & \Gamma_{212}\\ \Gamma_{122} & \Gamma_{222}\end{pmatrix}A^{-1}\begin{pmatrix}\Gamma_{111} & \Gamma_{211}\\ \Gamma_{121} & \Gamma_{221}\end{pmatrix} +$$

$$\begin{pmatrix}\Gamma_{112} & \Gamma_{212}\\ \Gamma_{122} & \Gamma_{222}\end{pmatrix}A^{-1}\begin{pmatrix}\Gamma_{111} & \Gamma_{211}\\ \Gamma_{121} & \Gamma_{221}\end{pmatrix}^T \qquad (11)$$

由(7)式、(10)式和(11)式,得

$$\frac{\partial}{\partial u_2}\begin{pmatrix}\Gamma_{111} & \Gamma_{211}\\ \Gamma_{121} & \Gamma_{221}\end{pmatrix} - \frac{\partial}{\partial u_1}\begin{pmatrix}\Gamma_{112} & \Gamma_{212}\\ \Gamma_{122} & \Gamma_{222}\end{pmatrix} -$$

$$\begin{pmatrix}\Gamma_{111} & \Gamma_{211}\\ \Gamma_{121} & \Gamma_{221}\end{pmatrix}A^{-1}\begin{pmatrix}\Gamma_{112} & \Gamma_{212}\\ \Gamma_{122} & \Gamma_{222}\end{pmatrix}^T +$$

$$\begin{pmatrix}\Gamma_{112} & \Gamma_{212}\\ \Gamma_{122} & \Gamma_{222}\end{pmatrix}A^{-1}\begin{pmatrix}\Gamma_{111} & \Gamma_{211}\\ \Gamma_{121} & \Gamma_{221}\end{pmatrix}^T =$$

$$\begin{pmatrix}0 & b_{11}b_{22} - b_{12}^2\\ b_{12}^2 - b_{11}b_{22} & 0\end{pmatrix} \qquad (12)$$

由(12)式两边矩阵的右上角对应元素相等,可得

$$\frac{\partial \Gamma_{211}}{\partial u_2} - \frac{\partial \Gamma_{212}}{\partial u_1} -$$

$$[(\Gamma_{111}g^{11} + \Gamma_{211}g^{21})\Gamma_{122} + (\Gamma_{111}g^{12} + \Gamma_{211}g^{22})\Gamma_{222}] +$$

$$[(\Gamma_{121}g^{11} + \Gamma_{221}g^{21})\Gamma_{112} + (\Gamma_{121}g^{12} + \Gamma_{221}g^{22})\Gamma_{212}] =$$

$$b_{11}b_{22} - b_{12}^2 = K(g_{11}g_{22} - g_{12}^2) \qquad (13)$$

再由

$$\begin{pmatrix}g^{11} & g^{12}\\ g^{21} & g^{22}\end{pmatrix} = \frac{1}{g}\begin{pmatrix}g_{22} & -g_{12}\\ -g_{12} & g_{11}\end{pmatrix}$$

可得

$$K = \frac{1}{g^2}\Big\{g\Big(\frac{\partial \Gamma_{211}}{\partial u_2} - \frac{\partial \Gamma_{212}}{\partial u_1}\Big) -$$

$$[(\Gamma_{111}g_{22} - \Gamma_{211}g_{12})\Gamma_{122} +$$

$$(-\Gamma_{111}g_{12} + \Gamma_{211}g_{11})\Gamma_{222}] +$$

$$[(\Gamma_{121}g_{22} - \Gamma_{221}g_{12})\Gamma_{112} +$$

$$(-\Gamma_{121}g_{12} + \Gamma_{221}g_{11})\Gamma_{212}]\Big\} \qquad (14)$$

利用 $\vec{r}_1 \cdot \vec{r}_1 = g_{11}$, $\vec{r}_1 \cdot \vec{r}_2 = g_{12}$, $\vec{r}_2 \cdot \vec{r}_2 = g_{22}$,可得

$$\vec{r}_1 \cdot \vec{r}_{11} = \frac{1}{2}\frac{\partial g_{11}}{\partial u_1},\ \vec{r}_1 \cdot \vec{r}_{12} = \frac{1}{2}\frac{\partial g_{11}}{\partial u_2}$$

$$\vec{r}_2 \cdot \vec{r}_{21} = \frac{1}{2}\frac{\partial g_{22}}{\partial u_1},\ \vec{r}_2 \cdot \vec{r}_{22} = \frac{1}{2}\frac{\partial g_{22}}{\partial u_2}$$

$$\vec{r}_2 \cdot \vec{r}_{11} = \frac{\partial g_{12}}{\partial u_1} - \frac{1}{2}\frac{\partial g_{11}}{\partial u_2},\ \vec{r}_1 \cdot \vec{r}_{22} =$$

$$\frac{\partial g_{12}}{\partial u_2} - \frac{1}{2}\frac{\partial g_{22}}{\partial u_1}$$

于是

$$\begin{pmatrix}\Gamma_{111} & \Gamma_{211}\\ \Gamma_{121} & \Gamma_{221}\end{pmatrix} =$$

$$\begin{pmatrix}\vec{r}_1 \cdot \vec{r}_{11} & \vec{r}_2 \cdot \vec{r}_{11}\\ \vec{r}_1 \cdot \vec{r}_{21} & \vec{r}_2 \cdot \vec{r}_{21}\end{pmatrix} = \begin{pmatrix}\frac{1}{2}\frac{\partial g_{11}}{\partial u_1} & \frac{\partial g_{12}}{\partial u_1} - \frac{1}{2}\frac{\partial g_{11}}{\partial u_2}\\ \frac{1}{2}\frac{\partial g_{11}}{\partial u_2} & \frac{1}{2}\frac{\partial g_{22}}{\partial u_1}\end{pmatrix}$$

$$\begin{pmatrix}\Gamma_{112} & \Gamma_{212}\\ \Gamma_{122} & \Gamma_{222}\end{pmatrix} =$$

$$\begin{pmatrix}\vec{r}_1 \cdot \vec{r}_{12} & \vec{r}_2 \cdot \vec{r}_{12}\\ \vec{r}_1 \cdot \vec{r}_{22} & \vec{r}_2 \cdot \vec{r}_{22}\end{pmatrix} = \begin{pmatrix}\frac{1}{2}\frac{\partial g_{11}}{\partial u_2} & \frac{1}{2}\frac{\partial g_{22}}{\partial u_1}\\ \frac{\partial g_{12}}{\partial u_2} - \frac{1}{2}\frac{\partial g_{22}}{\partial u_1} & \frac{1}{2}\frac{\partial g_{22}}{\partial u_2}\end{pmatrix} \qquad (15)$$

从而得出

$$\frac{\partial}{\partial u_2}\begin{pmatrix}\Gamma_{111} & \Gamma_{211}\\ \Gamma_{121} & \Gamma_{221}\end{pmatrix}-\frac{\partial}{\partial u_1}\begin{pmatrix}\Gamma_{112} & \Gamma_{212}\\ \Gamma_{122} & \Gamma_{222}\end{pmatrix}=\begin{pmatrix}0 & \frac{\partial^2 g_{12}}{\partial u_2\partial u_1}-\frac{1}{2}\frac{\partial^2 g_{11}}{\partial u_2\partial u_2}-\frac{1}{2}\frac{\partial^2 g_{22}}{\partial u_1\partial u_1}\\ -\left(\frac{\partial^2 g_{12}}{\partial u_2\partial u_1}-\frac{1}{2}\frac{\partial^2 g_{11}}{\partial u_2\partial u_2}-\frac{1}{2}\frac{\partial^2 g_{22}}{\partial u_1\partial u_1}\right) & 0\end{pmatrix}$$

$$\frac{\partial\Gamma_{211}}{\partial u_2}-\frac{\partial\Gamma_{212}}{\partial u_1}=\frac{\partial^2 g_{12}}{\partial u_2\partial u_1}-\frac{1}{2}\frac{\partial^2 g_{11}}{\partial u_2\partial u_2}-\frac{1}{2}\frac{\partial^2 g_{22}}{\partial u_1\partial u_1} \tag{16}$$

$$\frac{\partial\Gamma_{211}}{\partial u_2}-\frac{\partial\Gamma_{212}}{\partial u_1}=\frac{\partial}{\partial u_2}(\vec r_2\cdot\vec r_{11})-\frac{\partial}{\partial u_1}(\vec r_2\cdot\vec r_{12})=$$
$$(\vec r_{22}\cdot\vec r_{11}+\vec r_2\cdot\vec r_{112})-(\vec r_{21}\cdot\vec r_{12}+\vec r_2\cdot\vec r_{112})=$$
$$\vec r_{22}\cdot\vec r_{11}-\vec r_{21}\cdot\vec r_{12} \tag{17}$$

$$\frac{\partial}{\partial u_2}(\vec r_2\cdot\vec r_{11})-\frac{\partial}{\partial u_1}(\vec r_2\cdot\vec r_{12})=$$
$$\frac{\partial}{\partial u_2}\left(\frac{\partial g_{12}}{\partial u_1}-\frac{1}{2}\frac{\partial g_{11}}{\partial u_2}\right)-\frac{\partial}{\partial u_1}\left(\frac{1}{2}\frac{\partial g_{22}}{\partial u_1}\right)=$$
$$\frac{\partial^2 g_{12}}{\partial u_2\partial u_1}-\frac{1}{2}\frac{\partial^2 g_{11}}{\partial u_2\partial u_2}-\frac{1}{2}\frac{\partial^2 g_{22}}{\partial u_1\partial u_1}$$

所以有

$$\vec r_{22}\cdot\vec r_{11}-\vec r_{21}\cdot\vec r_{12}=\frac{\partial\Gamma_{211}}{\partial u_2}-\frac{\partial\Gamma_{212}}{\partial u_1}=$$
$$\frac{\partial^2 g_{12}}{\partial u_2\partial u_1}-\frac{1}{2}\frac{\partial^2 g_{11}}{\partial u_2\partial u_2}-\frac{1}{2}\frac{\partial^2 g_{22}}{\partial u_1\partial u_1} \tag{18}$$

在正交曲线坐标网下,可以求出系数矩阵,然后代入(14)式,就可给出高斯曲率的计算公式[1-6]。

6 高斯曲率是内蕴量的 Brioschi 公式表示

注意到

$$b_{11}=\vec n\cdot\vec r_{11}=\frac{(\vec r_1,\vec r_2,\vec r_{11})}{\sqrt{g}}$$
$$b_{12}=\vec n\cdot\vec r_{12}=\frac{(\vec r_1,\vec r_2,\vec r_{12})}{\sqrt{g}}$$
$$b_{22}=\vec n\cdot\vec r_{22}=\frac{(\vec r_1,\vec r_2,\vec r_{22})}{\sqrt{g}}$$

代入高斯曲率的计算公式[1-6,8]

$$K=\frac{b_{11}b_{22}-b_{12}^2}{g_{11}g_{22}-g_{12}^2}=\frac{b_{11}b_{22}-b_{12}^2}{g}=$$
$$\frac{1}{g^2}\left[(\vec r_1,\vec r_2,\vec r_{11})(\vec r_1,\vec r_2,\vec r_{22})-(\vec r_1,\vec r_2,\vec r_{12})^2\right]$$

利用行列式的转置性质和矩阵乘法性质,得

$$(\vec r_1,\vec r_2,\vec r_{11})(\vec r_1,\vec r_2,\vec r_{22})-(\vec r_1,\vec r_2,\vec r_{12})^2=$$
$$\begin{pmatrix}\vec r_1\\ \vec r_2\\ \vec r_{11}\end{pmatrix}(\vec r_1,\vec r_2,\vec r_{22})-\begin{pmatrix}\vec r_1\\ \vec r_2\\ \vec r_{12}\end{pmatrix}(\vec r_1,\vec r_2,\vec r_{12})=$$

$$\begin{vmatrix}\vec r_1\cdot\vec r_1 & \vec r_1\cdot\vec r_2 & \vec r_1\cdot\vec r_{22}\\ \vec r_2\cdot\vec r_1 & \vec r_2\cdot\vec r_2 & \vec r_2\cdot\vec r_{22}\\ \vec r_{11}\cdot\vec r_1 & \vec r_{11}\cdot\vec r_2 & \vec r_{11}\cdot\vec r_{22}\end{vmatrix}-$$
$$\begin{vmatrix}\vec r_1\cdot\vec r_1 & \vec r_1\cdot\vec r_2 & \vec r_1\cdot\vec r_{12}\\ \vec r_2\cdot\vec r_1 & \vec r_2\cdot\vec r_2 & \vec r_2\cdot\vec r_{12}\\ \vec r_{12}\cdot\vec r_1 & \vec r_{12}\cdot\vec r_2 & \vec r_{12}\cdot\vec r_{12}\end{vmatrix}=$$
$$\begin{vmatrix}g_{11} & g_{12} & \vec r_1\cdot\vec r_{22}\\ g_{12} & g_{22} & \vec r_2\cdot\vec r_{22}\\ \vec r_{11}\cdot\vec r_1 & \vec r_{11}\cdot\vec r_2 & \vec r_{11}\cdot\vec r_{22}\end{vmatrix}-$$
$$\begin{vmatrix}g_{11} & g_{12} & \vec r_1\cdot\vec r_{12}\\ g_{12} & g_{22} & \vec r_2\cdot\vec r_{12}\\ \vec r_{12}\cdot\vec r_1 & \vec r_{12}\cdot\vec r_2 & \vec r_{12}\cdot\vec r_{12}\end{vmatrix}=$$
$$\begin{vmatrix}g_{11} & g_{12} & \vec r_1\cdot\vec r_{22}\\ g_{12} & g_{22} & \vec r_2\cdot\vec r_{22}\\ \vec r_{11}\cdot\vec r_1 & \vec r_{11}\cdot\vec r_2 & \vec r_{11}\cdot\vec r_{22}-\vec r_{12}\cdot\vec r_{12}\end{vmatrix}-$$
$$\begin{vmatrix}g_{11} & g_{12} & \vec r_1\cdot\vec r_{12}\\ g_{12} & g_{22} & \vec r_2\cdot\vec r_{12}\\ \vec r_{12}\cdot\vec r_1 & \vec r_{12}\cdot\vec r_2 & 0\end{vmatrix}=$$
$$\begin{vmatrix}g_{11} & g_{12} & \Gamma_{122}\\ g_{12} & g_{22} & \Gamma_{222}\\ \Gamma_{111} & \Gamma_{211} & \vec r_{11}\cdot\vec r_{22}-\vec r_{12}\cdot\vec r_{12}\end{vmatrix}-$$
$$\begin{vmatrix}g_{11} & g_{12} & \Gamma_{112}\\ g_{12} & g_{22} & \Gamma_{212}\\ \Gamma_{112} & \Gamma_{212} & 0\end{vmatrix}$$

其中用到行列式按第三列展开计算的性质。利用

$$\vec r_{22}\cdot\vec r_{11}-\vec r_{21}\cdot\vec r_{12}=$$
$$\frac{\partial\Gamma_{211}}{\partial u_2}-\frac{\partial\Gamma_{212}}{\partial u_1}=\frac{\partial^2 g_{12}}{\partial u_2\partial u_1}-\frac{1}{2}\frac{\partial^2 g_{11}}{\partial u_2\partial u_2}-\frac{1}{2}\frac{\partial^2 g_{22}}{\partial u_1\partial u_1}$$

得到

$$K=\frac{1}{g^2}\left[\begin{vmatrix} g_{11} & g_{12} & \Gamma_{122} \\ g_{12} & g_{22} & \Gamma_{222} \\ \Gamma_{111} & \Gamma_{211} & \frac{\partial \Gamma_{211}}{\partial u_2}-\frac{\partial \Gamma_{212}}{\partial u_1} \end{vmatrix}-\begin{vmatrix} g_{11} & g_{12} & \Gamma_{112} \\ g_{12} & g_{22} & \Gamma_{212} \\ \Gamma_{112} & \Gamma_{212} & 0 \end{vmatrix}\right] \tag{19}$$

这就是高斯曲率内蕴定理的 Brioschi 表示公式[1-6]。

将(19)式中的两个行列式按第三列展开计算,可知 Brioschi 的表示公式(19)与(14)式是完全一样。

将(15)式、(16)式代入(19)式,得到 Brioschi 公式[1-6]

$$K=\frac{1}{g^2}\left[\begin{vmatrix} g_{11} & g_{12} & \frac{\partial g_{12}}{\partial u_2}-\frac{1}{2}\frac{\partial g_{22}}{\partial u_1} \\ g_{12} & g_{22} & \frac{1}{2}\frac{\partial g_{22}}{\partial u_2} \\ \frac{1}{2}\frac{\partial g_{11}}{\partial u_1} & \frac{\partial g_{12}}{\partial u_1}-\frac{1}{2}\frac{\partial g_{11}}{\partial u_2} & \frac{\partial^2 g_{12}}{\partial u_2\partial u_1}-\frac{1}{2}\frac{\partial^2 g_{11}}{\partial u_2\partial u_2}-\frac{1}{2}\frac{\partial^2 g_{22}}{\partial u_1\partial u_1} \end{vmatrix}-\begin{vmatrix} g_{11} & g_{12} & \frac{1}{2}\frac{\partial g_{11}}{\partial u_2} \\ g_{12} & g_{22} & \frac{1}{2}\frac{\partial g_{22}}{\partial u_1} \\ \frac{1}{2}\frac{\partial g_{11}}{\partial u_2} & \frac{1}{2}\frac{\partial g_{22}}{\partial u_1} & 0 \end{vmatrix}\right] \tag{20}$$

7　正交曲线坐标网下高斯曲率的简化计算公式的推导

当曲面 $\Sigma: \vec{r}=\vec{r}(u_1,u_2)$ 上的曲线坐标网是正交网时,$\vec{r}_1\cdot\vec{r}_2=g_{12}=0$,利用(18)式,得到

$$K=\frac{1}{(g_{11}g_{22})^2}\left[\begin{vmatrix} g_{11} & 0 & -\frac{1}{2}\frac{\partial g_{22}}{\partial u_1} \\ 0 & g_{22} & \frac{1}{2}\frac{\partial g_{22}}{\partial u_2} \\ \frac{1}{2}\frac{\partial g_{11}}{\partial u_1} & -\frac{1}{2}\frac{\partial g_{11}}{\partial u_2} & -\frac{1}{2}\frac{\partial^2 g_{11}}{\partial u_2\partial u_2}-\frac{1}{2}\frac{\partial^2 g_{22}}{\partial u_1\partial u_1} \end{vmatrix}-\begin{vmatrix} g_{11} & 0 & \frac{1}{2}\frac{\partial g_{11}}{\partial u_2} \\ 0 & g_{22} & \frac{1}{2}\frac{\partial g_{22}}{\partial u_1} \\ \frac{1}{2}\frac{\partial g_{11}}{\partial u_2} & \frac{1}{2}\frac{\partial g_{22}}{\partial u_1} & 0 \end{vmatrix}\right]=$$

$$\frac{1}{(g_{11}g_{22})^2}\Big\{g_{11}\Big[g_{22}\Big(-\frac{1}{2}\frac{\partial^2 g_{11}}{\partial u_2\partial u_2}-\frac{1}{2}\frac{\partial^2 g_{22}}{\partial u_1\partial u_1}\Big)+\frac{1}{2}\frac{\partial g_{11}}{\partial u_1}\frac{1}{2}\frac{\partial g_{22}}{\partial u_2}\Big]+g_{22}\frac{1}{2}\frac{\partial g_{11}}{\partial u_1}\frac{1}{2}\frac{\partial g_{22}}{\partial u_1}-\Big[-g_{11}\frac{1}{2}\frac{\partial g_{22}}{\partial u_1}\frac{1}{2}\frac{g_{22}}{\partial u_1}-g_{22}\frac{1}{2}\frac{\partial g_{11}}{\partial u_2}\frac{1}{2}\frac{\partial g_{11}}{\partial u_2}\Big]\Big\}=$$

$$-\Big[\frac{1}{2g_{11}g_{22}}\Big(\frac{\partial^2 g_{11}}{\partial u_2\partial u_2}+\frac{\partial^2 g_{22}}{\partial u_1\partial u_1}\Big)-\frac{1}{4(g_{11}g_{22})^2}\Big(g_{11}\frac{\partial g_{11}}{\partial u_2}\frac{\partial g_{22}}{\partial u_2}+g_{22}\frac{\partial g_{11}}{\partial u_1}\frac{\partial g_{22}}{\partial u_1}+g_{11}\frac{\partial g_{22}}{\partial u_1}\frac{\partial g_{22}}{\partial u_1}+g_{22}\frac{\partial g_{11}}{\partial u_2}\frac{\partial g_{11}}{\partial u_2}\Big)\Big] \tag{21}$$

另一方面,经过凑微分法,逐步推导,有

$$\frac{1}{2g_{11}g_{22}}\Big(\frac{\partial^2 g_{11}}{\partial u_2\partial u_2}+\frac{\partial^2 g_{22}}{\partial u_1\partial u_1}\Big)-\frac{1}{4(g_{11}g_{22})^2}\Big(g_{11}\frac{\partial g_{11}}{\partial u_2}\frac{\partial g_{22}}{\partial u_2}+g_{22}\frac{\partial g_{11}}{\partial u_1}\frac{\partial g_{22}}{\partial u_1}+g_{11}\frac{\partial g_{22}}{\partial u_1}\frac{\partial g_{22}}{\partial u_1}+g_{22}\frac{\partial g_{11}}{\partial u_2}\frac{\partial g_{11}}{\partial u_2}\Big)=$$

$$\frac{1}{\sqrt{g_{11}g_{22}}}\Big\{\frac{1}{2\sqrt{g_{11}g_{22}}}\Big(\frac{\partial^2 g_{11}}{\partial u_2\partial u_2}+\frac{\partial^2 g_{22}}{\partial u_1\partial u_1}\Big)-\frac{1}{4g_{11}g_{22}\sqrt{g_{11}g_{22}}}\Big[\frac{\partial g_{11}}{\partial u_2}\Big(g_{11}\frac{\partial g_{22}}{\partial u_2}+\frac{\partial g_{11}}{\partial u_2}g_{22}\Big)+\frac{\partial g_{22}}{\partial u_1}\Big(g_{11}\frac{\partial g_{22}}{\partial u_1}+\frac{\partial g_{11}}{\partial u_1}g_{22}\Big)\Big]\Big\}=$$

$$\frac{1}{2\sqrt{g_{11}g_{22}}}\Big(\frac{\partial^2 g_{11}}{\partial u_2\partial u_2}+\frac{\partial^2 g_{22}}{\partial u_1\partial u_1}\Big)-\frac{1}{4g_{11}g_{22}\sqrt{g_{11}g_{22}}}\Big[\frac{\partial g_{11}}{\partial u_2}\frac{\partial(g_{11}g_{22})}{\partial u_2}+\frac{\partial g_{22}}{\partial u_1}\frac{\partial(g_{11}g_{22})}{\partial u_1}\Big]=$$

$$\frac{1}{\sqrt{g_{11}g_{22}}}\Big\{\frac{1}{2\sqrt{g_{11}g_{22}}}\frac{\partial}{\partial u_2}\Big(\frac{\partial g_{11}}{\partial u_2}\Big)+\frac{\partial g_{11}}{\partial u_2}\frac{\partial}{\partial u_2}\Big(\frac{1}{2\sqrt{g_{11}g_{22}}}\Big)+\frac{1}{2\sqrt{g_{11}g_{22}}}\frac{\partial}{\partial u_1}\Big(\frac{\partial g_{22}}{\partial u_1}\Big)+\frac{\partial g_{22}}{\partial u_1}\frac{\partial}{\partial u_1}\Big(\frac{1}{2\sqrt{g_{11}g_{22}}}\Big)\Big\}=$$

$$\frac{1}{\sqrt{g_{11}g_{22}}}\Big[\frac{\partial}{\partial u_2}\Big(\frac{1}{2\sqrt{g_{11}g_{22}}}\frac{\partial g_{11}}{\partial u_2}\Big)+\frac{\partial}{\partial u_1}\Big(\frac{1}{2\sqrt{g_{11}g_{22}}}\frac{\partial g_{22}}{\partial u_1}\Big)\Big]=$$

$$\frac{1}{\sqrt{g_{11}g_{22}}}\Big[\frac{\partial}{\partial u_2}\Big(\frac{1}{\sqrt{g_{22}}}\frac{\partial\sqrt{g_{11}}}{\partial u_2}\Big)+\frac{\partial}{\partial u_1}\Big(\frac{1}{\sqrt{g_{11}}}\frac{\partial\sqrt{g_{22}}}{\partial u_1}\Big)\Big]$$

故有

$$K=-\frac{1}{\sqrt{g_{11}g_{22}}}\Big[\frac{\partial}{\partial u_2}\Big(\frac{1}{\sqrt{g_{22}}}\frac{\partial\sqrt{g_{11}}}{\partial u_2}\Big)+$$

$$\frac{\partial}{\partial u_1}\left(\frac{1}{\sqrt{g_{11}}}\frac{\partial\sqrt{g_{22}}}{\partial u_1}\right)\Bigg] \tag{22}$$

这就是正交曲线坐标网下高斯曲率的简化计算公式,具有重要的应用。

另一方面,需要直接计算

$$\frac{\partial}{\partial u_2}\left(\frac{1}{\sqrt{g_{22}}}\frac{\partial\sqrt{g_{11}}}{\partial u_2}\right)+\frac{\partial}{\partial u_1}\left(\frac{1}{\sqrt{g_{11}}}\frac{\partial\sqrt{g_{22}}}{\partial u_1}\right)$$

该式问题出现于利用刘维尔公式[2,6,10]

$$k_g=\frac{d\theta}{ds}-\frac{1}{2\sqrt{g_{11}g_{22}}}\frac{\partial g_{11}}{\partial u_2}\frac{du_1}{ds}+\frac{1}{2\sqrt{g_{11}g_{22}}}\frac{\partial g_{22}}{\partial u_1}\frac{du_2}{ds}$$

证明高斯—波涅公式的过程中[2-6]。

直接计算可得

$$\frac{\partial}{\partial u_2}\left(\frac{1}{\sqrt{g_{22}}}\frac{\partial\sqrt{g_{11}}}{\partial u_2}\right)+\frac{\partial}{\partial u_1}\left(\frac{1}{\sqrt{g_{11}}}\frac{\partial\sqrt{g_{22}}}{\partial u_1}\right)=$$

$$\sqrt{g_{11}g_{22}}\Bigg[\frac{1}{2g_{11}g_{22}}\left(\frac{\partial^2 g_{11}}{\partial u_2\partial u_2}+\frac{\partial^2 g_{22}}{\partial u_1\partial u_1}\right)-$$

$$\frac{1}{4(g_{11}g_{22})^2}\Bigg(g_{11}\frac{\partial g_{11}}{\partial u_2}\frac{\partial g_{22}}{\partial u_2}+$$

$$g_{22}\frac{\partial g_{11}}{\partial u_1}\frac{\partial g_{22}}{\partial u_1}+g_{11}\frac{\partial g_{22}}{\partial u_1}\frac{\partial g_{22}}{\partial u_1}+g_{22}\frac{\partial g_{11}}{\partial u_2}\frac{\partial g_{11}}{\partial u_2}\Bigg)\Bigg] \tag{23}$$

由(21)式和(23)式,得

$$K=-\frac{1}{\sqrt{g_{11}g_{22}}}\Bigg[\frac{\partial}{\partial u_2}\left(\frac{1}{\sqrt{g_{22}}}\frac{\partial\sqrt{g_{11}}}{\partial u_2}\right)+$$

$$\frac{\partial}{\partial u_1}\left(\frac{1}{\sqrt{g_{11}}}\frac{\partial\sqrt{g_{22}}}{\partial u_1}\right)\Bigg]$$

将(22)式改写为如下形式

$$K=-\frac{1}{\sqrt{g_{11}g_{22}}}\Bigg[\frac{\partial}{\partial u_2}\left(\frac{1}{2\sqrt{g_{11}g_{22}}}\frac{\partial g_{11}}{\partial u_2}\right)+$$

$$\frac{\partial}{\partial u_1}\left(\frac{1}{2\sqrt{g_{11}g_{22}}}\frac{\partial g_{22}}{\partial u_1}\right)\Bigg]=$$

$$-\frac{1}{\sqrt{g}}\left[\frac{\partial}{\partial u_2}\left(\frac{1}{\sqrt{g}}\Gamma_{112}\right)+\frac{\partial}{\partial u_1}\left(\frac{1}{\sqrt{g}}\Gamma_{221}\right)\right] \tag{24}$$

利用

$$\begin{pmatrix}\Gamma_{1j}^1 & \Gamma_{1j}^2\\ \Gamma_{2j}^1 & \Gamma_{2j}^2\end{pmatrix}=\begin{pmatrix}\Gamma_{11j} & \Gamma_{21j}\\ \Gamma_{12j} & \Gamma_{22j}\end{pmatrix}A^{-1}$$,及 $\vec{r}_1\cdot\vec{r}_2=g_{12}=0$,

得

$$\begin{pmatrix}\Gamma_{11}^1 & \Gamma_{11}^2\\ \Gamma_{21}^1 & \Gamma_{21}^2\end{pmatrix}=\begin{pmatrix}\Gamma_{111} & \Gamma_{211}\\ \Gamma_{121} & \Gamma_{221}\end{pmatrix}A^{-1}=$$

$$\begin{pmatrix}\Gamma_{111} & -\Gamma_{112}\\ \Gamma_{112} & \Gamma_{221}\end{pmatrix}\begin{pmatrix}\frac{1}{g_{11}} & 0\\ 0 & \frac{1}{g_{22}}\end{pmatrix}=$$

$$\begin{pmatrix}\frac{1}{g_{11}}\Gamma_{111} & -\frac{1}{g_{22}}\Gamma_{112}\\ \frac{1}{g_{11}}\Gamma_{112} & \frac{1}{g_{22}}\Gamma_{221}\end{pmatrix} \tag{25}$$

由此得出

$$\Gamma_{11}^2=-\frac{1}{g_{22}}\Gamma_{112},\ \Gamma_{12}^2=\Gamma_{21}^2=\frac{1}{g_{22}}\Gamma_{221} \tag{26}$$

将(26)式代入(24)式,得

$$K=-\frac{1}{\sqrt{g}}\left[\frac{\partial}{\partial u_2}\left(\frac{1}{\sqrt{g}}\Gamma_{112}\right)+\frac{\partial}{\partial u_1}\left(\frac{1}{\sqrt{g}}\Gamma_{221}\right)\right]=$$

$$-\frac{1}{\sqrt{g}}\left[\frac{\partial}{\partial u_2}\left(-\frac{g_{22}}{\sqrt{g}}\Gamma_{11}^2\right)+\frac{\partial}{\partial u_1}\left(\frac{g_{22}}{\sqrt{g}}\Gamma_{12}^2\right)\right]=$$

$$\frac{1}{\sqrt{g}}\left[\frac{\partial}{\partial u_2}\left(\frac{\sqrt{g}}{g_{11}}\Gamma_{11}^2\right)-\frac{\partial}{\partial u_1}\left(\frac{\sqrt{g}}{g_{11}}\Gamma_{12}^2\right)\right] \tag{27}$$

类似地

$$\begin{pmatrix}\Gamma_{12}^1 & \Gamma_{12}^2\\ \Gamma_{22}^1 & \Gamma_{22}^2\end{pmatrix}=\begin{pmatrix}\Gamma_{112} & \Gamma_{212}\\ \Gamma_{122} & \Gamma_{222}\end{pmatrix}A^{-1}=$$

$$\begin{pmatrix}\Gamma_{112} & \Gamma_{221}\\ -\Gamma_{221} & \Gamma_{222}\end{pmatrix}\begin{pmatrix}\frac{1}{g_{11}} & 0\\ 0 & \frac{1}{g_{22}}\end{pmatrix}=$$

$$\begin{pmatrix}\frac{1}{g_{11}}\Gamma_{112} & \frac{1}{g_{22}}\Gamma_{221}\\ -\frac{1}{g_{11}}\Gamma_{221} & \frac{1}{g_{22}}\Gamma_{222}\end{pmatrix}$$

$$\Gamma_{12}^1=\frac{1}{g_{11}}\Gamma_{112},\ \Gamma_{22}^1=-\frac{1}{g_{11}}\Gamma_{221} \tag{28}$$

将(28)式代入(24)式,得

$$K=-\frac{1}{\sqrt{g}}\left[\frac{\partial}{\partial u_2}\left(\frac{1}{\sqrt{g}}\Gamma_{112}\right)+\frac{\partial}{\partial u_1}\left(\frac{1}{\sqrt{g}}\Gamma_{221}\right)\right]=$$

$$-\frac{1}{\sqrt{g}}\left[\frac{\partial}{\partial u_2}\left(\frac{g_{11}}{\sqrt{g}}\Gamma_{12}^1\right)-\frac{\partial}{\partial u_1}\left(\frac{g_{11}}{\sqrt{g}}\Gamma_{22}^1\right)\right]=$$

$$\frac{1}{\sqrt{g}}\left[\frac{\partial}{\partial u_1}\left(\frac{\sqrt{g}}{g_{22}}\Gamma_{22}^1\right)-\frac{\partial}{\partial u_2}\left(\frac{\sqrt{g}}{g_{22}}\Gamma_{12}^1\right)\right] \tag{29}$$

对(27)式和(29)式,这里给出导出发现过程。

在一般坐标曲线网下,直接验证[1-2,7,9],成立

$$K=\frac{1}{\sqrt{g}}\left[\frac{\partial}{\partial u_2}\left(\frac{\sqrt{g}}{g_{11}}\Gamma_{11}^2\right)-\frac{\partial}{\partial u_1}\left(\frac{\sqrt{g}}{g_{11}}\Gamma_{12}^2\right)\right]$$

$$K = \frac{1}{\sqrt{g}}\left[\frac{\partial}{\partial u_1}\left(\frac{\sqrt{g}}{g_{22}}\Gamma_{22}^1 \right) - \frac{\partial}{\partial u_2}\left(\frac{\sqrt{g}}{g_{22}}\Gamma_{12}^1 \right) \right]$$

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Derivations with Several forms of Intrinsic Formulas of Gaussian Curvature

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Abstract: The expression of intrinsic formula of Gaussian curvature on curved surface is considered. The direct explicit formula of the intrinsic quantities expressed by Gaussian curvature is derived by means of the matrix expression of curved surface fundamental equation. This intrinsic formula is evidently in accord with Brioschi formula. The derivation of Gaussian curvature simplified formula is demonstrated, which discloses the discovery process of Gaussian curvature implicit formula.

Key words: curved surface fundamental equation; curved surface constitutive equation; Gaussian curvature; intrinsic formula; Brioschi formula