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GARCH 模型的经验似然估计

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摘　要:以计量经济学中 ARCH 模型族为背景,对 GARCH 模型进行讨论和研究。讨论了基于 GED 分布的 β-ARCH 模型和 GARCH 模型的经验极大似然估计的求解方法,得到了相应的平稳模型的大样本性质定理。

关键词:自回归条件异方差(ARCH)模型;广义自回归条件异方差(GARCH)模型;经验似然估计

中图分类号:O213　　**文献标志码**:A

早在 1975 年,Thomas 和 Grunkemier 在建立截尾数据下生存概率的区间估计时已经萌发了经验似然的思想。1988 年,Owen[1-4] 提出了经验似然的概念,在约束条件下极大化经验似然函数后就得到了经验似然估计。这其中,约束条件是通过重新分配每个观测值上的权重而实现它对参数估计的影响。研究成果表明,经验似然估计具有类似于 Fisher 提出的参数似然法那样的优良性质。

1　经验似然估计

1.1　$\beta-ARCH$ 模型的经验似然估计

时间序列 $\{X_t\}$ 满足下列模型

$$\begin{cases} X_t = \varepsilon_t \\ \varepsilon_t = \eta_t h_t^{\frac{1}{2}} \\ h_t = \alpha_0 + \alpha_1 |\varepsilon_{t-1}|^{\gamma\beta} + \cdots + \alpha_q |\varepsilon_{t-q}|^{\gamma\beta} \end{cases}$$

考虑这样的问题,对于样本 $X_1, X_2, \cdots, X_n$ 令 $p_t = P(X = X_t)(t = 1,2,\cdots,n)$, 则有 $\sum_{t=1}^{n} p_t = 1$, 此外矩约束条件为: $E(X) = \sum_{t=1}^{n} p_t X_t = 0$, 此时,最优解是在约束条件之下使得经验似然函数 $L_n(\alpha) = \prod_{t=1}^{n} p_t$ 达到最大值的 $\hat{\alpha}$。

优化对象

$$R_n(\alpha) = \ln L_n(\alpha) = \sum_{t=1}^{n} \ln p_t(\alpha)$$

在此约束条件下,取拉格朗日函数为

$$G = \sum_{t=1}^{n} \ln p_t + \mu(1 - \sum_{t=1}^{n} p_t) - n\lambda^T \sum_{t=1}^{n} p_t g_t(\alpha)$$

其中,μ,λ 为拉格朗日乘子对各变量求导,并应用 KKT 条件,最优解必然满足下列条件

$$\frac{\partial G}{\partial p_t} = \frac{1}{p_t} - \mu - n\lambda^T g_t(\alpha) = 0$$

解得

$$p_t = \frac{1}{\mu + n\lambda^T g_t(\alpha)}$$

由假设

$$\sum_{t=1}^{n} p_t = \sum_{t=1}^{n} \frac{1}{\mu + n\lambda^T g_t(\alpha)} = 1$$

且

$$\sum_{t=1}^{n} n\lambda^T g_t(\alpha) = 0$$

解得

$$\mu = n$$

代入方程解得

$$p_t = \frac{1}{n[1 + \lambda^T g_t(\alpha)]}$$

已知 $0 < p_t < 1$, 可知

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$$1+\lambda^{T}g_{t}(\alpha)>\frac{1}{n}$$

拉格朗日乘子是下述方程的解

$$\sum_{t=1}^{n}p_{t}g_{t}(\alpha)=\frac{1}{n}\sum_{t=1}^{n}\frac{g_{t}(\alpha)}{1+\lambda^{T}g_{t}(\alpha)}=0$$

另外,我们注意到

$$\frac{\partial}{\partial\lambda}\left\{\frac{1}{n}\sum_{t=1}^{n}\left(\frac{g_{t}(\alpha)}{1+\lambda^{T}g_{t}(\alpha)}\right)\right\}=$$

$$-\frac{1}{n}\sum_{t=1}^{n}\frac{g_{t}(\alpha)g_{t}^{T}(\alpha)}{[1+\lambda^{T}g_{t}(\alpha)]^{2}}$$

根据隐函数存在定理,λ 可以视为 α 的连续可微函数,且 λ 的取值与样本容量相关,不妨记作 $\lambda_{n}(\alpha)$。对于给定的 α,对应的经验似然函数为:

$$L_{n}(\alpha)=\prod_{t=1}^{n}p_{t}=\prod_{t=1}^{n}\frac{1}{n[1+\lambda^{T}g_{t}(\alpha)]}=$$

$$\left(\frac{1}{n}\right)^{n}\prod_{t=1}^{n}[1+\lambda^{T}g_{t}(\alpha)]^{-1}$$

则相应的对数经验似然函数为:

$$\log L_{n}(\alpha)=-n\log n-\sum_{t=1}^{n}\log[1+\lambda^{T}g_{t}(\alpha)]$$

这样,经验极大似然估计(MELE)问题就被解决了。

1.2 *GARCH* 模型的经验似然估计

考虑 *GARCH* 模型

$$\begin{cases}X_{t}=\varepsilon_{t}\sqrt{h_{t}}\\ h_{t}=\varpi+\sum_{i=1}^{r}\alpha_{i}X_{t-i}^{2}+\sum_{j=1}^{s}\beta_{i}h_{t-j}\end{cases}$$

其中 $\varpi>0,\alpha_{i},\beta_{j}\geqslant0$,$\varepsilon_{t}$ 是独立同分布的随机变量,期望值为 0,方差为 1;定义 $\lambda=(\varpi,\alpha_{1},\cdots,\alpha_{p},\beta_{1},\cdots,\beta_{j})^{T}\subset\Theta$,其中 Θ 是 R^{p+q+1} 上的完备集合。

若满足以上讨论的 *GARCH* 模型严格平稳,即有

$$\lambda\in\Theta,\varpi>0,\alpha_{i},\beta_{j}\geqslant0,\sum_{i=1}^{p}\alpha i+\sum_{j=i}^{q}\beta_{j}<1,EX_{t}^{2}<\infty$$

设定 ε_{t} 服从高斯分布,则

$$l_{t}(\lambda)=-\frac{1}{2}\log h_{t}(\lambda)-\frac{X_{t}^{2}}{2h_{t}(\lambda)}L(\lambda)=\sum_{t=1}^{n}l_{t}(\lambda)$$

现假定 ε_{t} 服从广义误差分布,基于广义误差分布的概率密度函数形式,为简化符号,令

$$a(r)=\frac{\gamma}{\lambda2^{1+\frac{1}{\gamma}}\Gamma(\frac{1}{\gamma})},\ b(r)=\frac{1}{2}\lambda^{-r}$$

这样,$X\sim GED(0,\sigma^{r},\gamma)$,则密度函数可以写成

$$f(x;r)=a(r)\frac{1}{\sigma}e^{\{-b(r)\frac{|x|^{\gamma}}{\lambda}\}}$$

容易计算,

$$EX=\int_{-\infty}^{+\infty}x\,a(r)\frac{1}{\sigma}e^{\{-b(r)\frac{|x|^{\gamma}}{\lambda}\}}dx=0\quad E|X|^{r}=$$

$$\int_{-\infty}^{+\infty}|x|^{r}a(r)\frac{1}{\sigma}e^{\{-b(r)\frac{|x|^{\gamma}}{\lambda}\}}dx=$$

$$2\sigma^{r}\int_{0}^{+\infty}a(r)\left(\frac{x}{\sigma}\right)^{r}e^{\{-b(r)(\frac{x}{\sigma})^{r}\}}d\frac{x}{\sigma}=\frac{\sigma^{r}}{rb(r)}$$

综合上式,

$$l_{t}(\lambda,r)=-\frac{|X|^{r}}{2}+\log\frac{a(r)}{h_{t}(\lambda)^{\frac{1}{2}}}=$$

$$\frac{\gamma}{\lambda2^{1+\frac{1}{\gamma}}\Gamma(\frac{1}{\gamma})h_{t}(\lambda)^{\frac{1}{2}}}$$

且

$$L(\lambda)=\sum_{t=1}^{n}l_{t}(\lambda)=\sum_{t=1}^{n}\frac{\gamma}{\lambda2^{1+\frac{1}{\gamma}}\Gamma(\frac{1}{\gamma})h_{t}(\lambda)^{\frac{1}{2}}}$$

对于独立同分布的随机变量 $y_{1},\cdots,y_{n}$,y_{i} 的分布函数 F 是基于经验分布函数 $F_{n}=\sum_{i=1}^{n}p_{i}I_{\{y_{i}\leqslant x\}}$,且满足 $\sum_{t=1}^{n}p_{t}=1$。

Owen[4-5] 定义了经验似然比。令 $R(F)=\prod_{i=1}^{n}np_{i}$,$p_{i}=dF(y_{i})=P(Y=y_{i})$,$\lambda$ 满足使得函数 $g(x_{i},\lambda)=0$ 且

$$R(\lambda)=\sup\{\prod_{i=1}^{n}np_{i}:p_{i}\geqslant0,\sum_{i,=1}^{n}p_{i}=1$$

$$\sum_{i=1}^{n}p_{i}g(x_{i},\lambda)=0\}$$

可以通过拉格朗日乘子法求得最大的 $R(\lambda)$。

定义拉格朗日函数为

$$L(\lambda)=\sum_{t=1}^{n}\log p_{i}+a(1-\sum_{t=1}^{n}p_{i})-nb^{T}\sum_{t=1}^{n}p_{i}g_{i}(r,\lambda)$$

其中,a,b 为拉格朗日乘子,则经验似然函数为

$$L_{E}(\lambda)=\sum_{i=1}^{n}\log[1+b^{T}(\lambda)g(x_{i},\lambda)]$$

上述过程中求得的 $\hat{\lambda}_{n}$ 能够使得 $L_{E}(\lambda)$ 取得最大值点,经验最大似然估计(MELE)问题被解决了。

2 *GARCH* 模经验似然估计的大样本性质

2.1 $\beta-ARCH$ 模型经验似然估计的大样本性质

引理 1 当 $n\to\infty$,而且当 $\hat{\alpha}_{n}$ 是内点时,它满足 $Q_{1n}(\hat{\alpha}_{n},\hat{\lambda}_{n})=0,Q_{2n}(\hat{\alpha}_{n},\hat{\lambda}_{n})=0$,其中,

$$Q_{1n}=\frac{1}{n}\sum_{t=1}^{n}\frac{g_{t}(\lambda)}{1+\lambda_{n}^{T}g_{t}(\alpha)},Q_{2n}=$$

$$\frac{1}{n}\sum_{t=1}^{n}\frac{1}{1+\lambda_{n}^{T}(\alpha)g_{t}(\lambda)}\backslash\{\frac{\partial g_{t}(\alpha)}{\partial\alpha}\backslash\}^{T}\lambda_{n}(\alpha)$$

证明 见文献[6]。

定理 1 设矩阵

$$S_n=\begin{pmatrix}\frac{\partial Q_{1n}(\lambda,b)}{\partial\lambda^T} & \frac{\partial Q_{1n}(\lambda,b)}{\partial\alpha}\\ \frac{\partial Q_{2n}(\lambda,b)}{\partial\lambda^T} & \frac{\partial Q_{2n}(\lambda,b)}{\partial\alpha}\end{pmatrix}=$$

$$\begin{pmatrix}S_{11} & S_{12}\\ S_{21} & S_{22}\end{pmatrix},\ S_n^{-1}=\begin{bmatrix}T_{11} & T_{12}\\ T_{21} & T_{22}\end{bmatrix}$$

令 $U=T_{11}(-S_{11})T_{11}^T, V=T_{21}(-S_{11})T_{21}^T$,则有

$$\sqrt{n}(\hat\lambda_n-0)\xrightarrow{L}N(0,U),\sqrt{n}(\hat\alpha_n-\alpha^{(0)})\xrightarrow{L}N(0,V)$$

其中 $Q_{1n}(\lambda,b)$, $Q_{2n}(\lambda,b)$ 如引理 1 所述。

证明　首先将 $Q_{1n}((\hat\alpha_n,\hat\lambda_n))$, $Q_{2n}(\hat\lambda_n,b(\hat\lambda_n))$ 在 $(\alpha^{(0)},0)$ 处进行泰勒展开。

$$Q_{1n}(\hat\lambda_n,\hat b_n)=Q_{1n}(\alpha^{(0)},0)+$$

$$\frac{\partial Q_{1n}(\alpha,\lambda)}{\partial\alpha}\Big|_{\alpha=\alpha^{(0)},\lambda=0}(\hat\alpha_n-\alpha^{(0)})+$$

$$\frac{\partial Q_{1n}(\alpha,\lambda)}{\partial\lambda^T}\Big|_{\alpha=\alpha^{(0)},\lambda=0}(\hat\lambda_n-0)+O_p(\delta_n)=$$

$$0Q_{2n}(\hat\lambda_n,\hat b_n)=Q_{2n}(\alpha^{(0)},0)+$$

$$\frac{\partial Q_{2n}(\alpha,\lambda)}{\partial\alpha}\Big|_{\alpha=\alpha^{(0)},\lambda=0}(\hat\alpha_n-\alpha^{(0)})+$$

$$\frac{\partial Q_{2n}(\alpha,\lambda)}{\partial\lambda^T}\Big|_{\alpha=\alpha^{(0)},\lambda=0}(\hat\lambda_n-0)+O_p(\delta_n)=0$$

其中,

$$Q_{1n}(\alpha^{(0)},0)=\frac{1}{n}\sum_t g_t(\alpha^{(0)}),\ Q_{2n}(\alpha^{(0)},0)=0$$

$$\frac{\partial Q_{1n}(\alpha,\lambda)}{\partial\alpha}\Big|_{\alpha=\alpha^{(0)},\lambda=0}=\frac{1}{n}\sum_t\frac{\partial g_t(\alpha^{(0)})}{\partial\alpha}$$

$$\frac{\partial Q_{2n}(\alpha,\lambda)}{\partial\alpha}\Big|_{\alpha=\alpha^{(0)},\lambda=0}=0\ \frac{\partial Q_{1n}(\alpha,\lambda)}{\partial\lambda^T}\Big|_{\alpha=\alpha^{(0)},\lambda=0}=$$

$$-\frac{1}{n}\sum_t g_t(\alpha^{(0)})g_t^T(\alpha^{(0)})$$

$$\frac{\partial Q_{2n}(\alpha,\lambda)}{\partial\lambda^T}\Big|_{\alpha=\alpha^{(0)},\lambda=0}=\frac{1}{n}\sum_t\left(\frac{\partial g_t(\alpha^{(0)})}{\partial\alpha}\right)^T$$

则上式即为

$$\begin{pmatrix}\hat\lambda_n-0\\ \hat\alpha_n-\alpha^{(0)}\end{pmatrix}=$$

$$S_n^{-1}\begin{pmatrix}-Q_{1n}(\alpha^{(0)},0)+o_p(n^{-\frac{1}{2}})\\ o_p(n^{-\frac{1}{2}})\end{pmatrix}$$

其中:

$$S_n=\begin{pmatrix}\frac{\partial Q_{1n}(\lambda,b)}{\partial\lambda^T} & \frac{\partial Q_{1n}(\lambda,b)}{\partial\alpha}\\ \frac{\partial Q_{2n}(\lambda,b)}{\partial\lambda^T} & \frac{\partial Q_{2n}(\lambda,b)}{\partial\alpha}\end{pmatrix}$$

且满足

$$S_n=\begin{pmatrix}S_{11} & S_{12}\\ S_{21} & S_{22}\end{pmatrix}=$$

$$\begin{pmatrix}E[g_t(\alpha^{(0)})g_t^T(\alpha^{(0)})] & E(\frac{\partial g_t(\alpha^{(0)})}{\partial\alpha})\\ E(\frac{\partial g_t(\alpha^{(0)})}{\partial\alpha})^T & 0\end{pmatrix}$$

$$T_n=\begin{bmatrix}T_{11} & T_{12}\\ T_{21} & T_{22}\end{bmatrix}$$

与此同时,需要求解出 S_n 的逆矩阵。

设 S_n 的逆矩阵为 T_n,满足

$$\begin{bmatrix}T_{11} & T_{12}\\ T_{21} & T_{22}\end{bmatrix}\begin{bmatrix}S_{11} & S_{12}\\ S_{21} & S_{22}\end{bmatrix}=\begin{bmatrix}E_m & 0\\ 0 & E_n\end{bmatrix}$$

联立得方程组:

$$\begin{cases}T_{11}S_{11}+T_{12}S_{21}=E_m\\ T_{11}S_{12}+T_{12}S_{22}=0\\ T_{21}S_{11}+T_{22}S_{21}=0\\ T_{21}S_{12}+T_{22}S_{22}=E_n\end{cases}$$

化简:

$$\begin{cases}-T_{12}S_{22}S_{12}^{-1}S_{11}+T_{12}S_{21}=E_m\\ -T_{22}S_{21}S_{11}^{-1}S_{12}+T_{22}S_{22}=E_n\end{cases}$$

解得

$$T_{12}=(S_{21}-S_{22}S_{11}S_{12}^{-1})^{-1}T_{22}=(S_{22}-S_{21}S_{11}^{-1}S_{12})^{-1}$$

$$T_{11}=-(S_{21}-S_{22}S_{11}S_{12}^{-1})^{-1}S_{22}S_{12}^{-1}$$

$$T_{21}=-(S_{22}-S_{21}S_{12}S_{11}^{-1})^{-1}S_{21}S_{11}^{-1}$$

这里,为简化起见,令

$$S_{22.1}=S_{22}-S_{21}S_{11}^{-1}S_{12}=-S_{21}S_{11}^{-1}S_{12}$$

进而得到

$$\sqrt{n}\begin{pmatrix}\check\lambda_n-0\\ \check\alpha_n-\alpha^{(0)}\end{pmatrix}=S_n^{-1}\begin{pmatrix}-\sqrt{n}Q_{1n}(\alpha^{(0)},0)+o_p(1)\\ o_p(1)\end{pmatrix}$$

令 $n\to\infty$,由于 $\sqrt{n}Q_{1n}(\alpha^{(0)},0)$ 的渐近正态性,得

$$\sqrt{n}\begin{pmatrix}\hat\lambda_n-0\\ \hat\alpha_n-\alpha^{(0)}\end{pmatrix}\xrightarrow{L}N(0,W)$$

相应的

$$\begin{bmatrix}T_{11} & T_{12}\\ T_{21} & T_{22}\end{bmatrix}\begin{pmatrix}-S_{11} & 0\\ 0 & 0\end{pmatrix}\begin{bmatrix}T_{11}^T & T_{12}^T\\ T_{21}^T & T_{22}^T\end{bmatrix}=$$

$$\begin{pmatrix}T_{11}(-S_{11})T_{11}^T & T_{11}(-S_{11})T_{21}^T\\ T_{21}(-S_{11})T_{11}^T & T_{21}(-S_{11})T_{21}^T\end{pmatrix}$$

其中,

$$V=T_{21}(-S_{11})T_{21}^T$$

$$U=T_{11}(-S_{11})T_{11}^T$$

特别的

$$\sqrt{n}(\hat\lambda_n - 0) = -(S_{11}^{-1} + S_{11}^{-1}S_{12}S_{22.1}^{-1}S_{21}S_{11}^{-1}) \times \sqrt{n}Q_{1n}(\alpha^{(0)},0) + o_p(1)$$

$$\sqrt{n}(\check\alpha_n - \alpha^{(0)}) = S_{22.1}^{-1}S_{21}S_{11}^{-1}\sqrt{n}Q_{1n}(\alpha^{(0)},0) + o_p(1)$$

得到

$$\sqrt{n}(\hat\lambda_n - 0) \xrightarrow{L} N(0,U)$$

$$\sqrt{n}(\hat\alpha_n - \alpha^{(0)}) \xrightarrow{L} N(0,V)$$

2.2 GARCH 模型经验似然估计的大样本性质

定理 2 若 GARCH 模型严格平稳，且 $EX_t^2 < \infty$，$\lambda \in \Theta, \varpi > 0, \alpha_i, \beta_j \geqslant 0, \sum_{i=1}^{p}\alpha i + \sum_{j=i}^{q}\beta_j < 1$ 对于所有的 $\delta > 0$ 满足 $E|\varepsilon_t|^{4+\delta} < \infty$，对于讨论的 $\hat\lambda_n(\theta_0)$ [7-10]，当 $n \to \infty$ 时，使得 $l_n(r;\theta_0;\hat\lambda_n(\theta_0)) \xrightarrow{d} \chi^2(1)$。

证明 定义 $g_t(\lambda) = g_t(\theta_0,\lambda)$，令

$$Q_{1n}(\lambda,b) = \frac{1}{n}\sum_{t=1}^{n}\frac{g_t(\lambda)}{1+b^Tg_t(\lambda)},\ Q_{2n}(\lambda,b) = \frac{1}{n}\sum_{t=1}^{n}\frac{1}{1+b^Tg_t(\lambda)}\{\frac{\partial g_t(\lambda)}{\partial\lambda^T}\}^Tb$$

由于

$$\frac{\partial \ln(r;\theta_0,\lambda)}{\partial\lambda}|_{\lambda=\lambda_n(\theta_0)} = \sum_{t=1}^{n}\frac{2}{1+b^Tg_t(\lambda)}\{\frac{\partial g_t(\lambda)}{\partial\lambda^T}\}^Tb(\lambda)|_{\lambda=\lambda_n(\theta_0)} = 0$$

易得

$$Q_{2n}(\hat\lambda_n,b(\hat\lambda_n)) = 0$$

$$\frac{\partial Q_{2n}(\lambda,b)}{\partial\lambda_i} = \frac{1}{n}\sum_{t=1}^{n}\{\frac{1}{1+b^Tg_t(\lambda)}\frac{\partial}{\partial\lambda_i}(\frac{\partial g_t(\lambda)}{\partial\lambda^T})^Tb - (\frac{\partial g_t(\lambda)}{\partial\lambda^T})^Tb\frac{b^T\frac{\partial g_t(\lambda)}{\partial\lambda_i}}{\{1+b^Tg_t(\lambda)\}^2}\}\ \frac{\partial Q_{2n}(\lambda,b)}{\partial\lambda_i} \leqslant$$

$$\frac{b}{1-\max|\gamma_t|}\frac{1}{n}\sum_{t=1}^{n}\frac{\partial}{\partial\lambda_i}(\frac{\partial g_t(\lambda)}{\partial\lambda^T}) + \frac{b^2\max_{1\leqslant t\leqslant n}\frac{\partial g_t(\lambda)}{\partial\lambda^T}}{(1-\max_{1\leqslant t\leqslant n}|\gamma_t|)^2}\frac{1}{n}\sum_{t=1}^{n}\frac{\partial g_t(\lambda)}{\partial\lambda^T}$$

则

$$\frac{\partial Q_{2n}(\lambda,b)}{\partial\lambda^T} \xrightarrow{p} 0$$

同理

$$\frac{\partial Q_{2n}(\lambda,b)}{\partial b^T} = \frac{1}{n}\sum_{t=1}^{n}\{\frac{1}{1+b^Tg_t(\lambda)}(\frac{\partial g_t(\lambda)}{\partial\lambda^T})^T - (\frac{\partial g_t(\lambda)}{\partial\lambda^T})^Tb\frac{g_t(\lambda)}{\{1+b^Tg_t(\lambda)\}^2}\}$$

定义 Ω 是实数域上的矩阵，且在 $R^{(p+q+2)(p+q+1)}$ 上，使得

$$\frac{\partial Q_{2n}(\lambda,b)}{\partial b^T} - \Omega^T \leqslant \frac{\max_{1\leqslant t\leqslant n}|\gamma_t|}{1-\max_{1\leqslant t\leqslant n}|\gamma_t|}\frac{1}{n}\sum_{t=1}^{n}\frac{\partial g_t(\lambda)}{\partial\lambda^T} + \frac{1}{n}\sum_{t=1}^{n}(\frac{\partial g_t(\lambda)}{\partial\lambda^T})^T - \Omega^T + \frac{\max_{1\leqslant t\leqslant n}|\gamma_t|}{(1-\max_{1\leqslant t\leqslant n}|\gamma_t|)^2}\frac{1}{n}\sum_{t=1}^{n}\frac{\partial g_t(\lambda)}{\partial\lambda^T}$$

且易推得

$$\frac{\partial Q_{1n}(\lambda,b)}{\partial\lambda^T} = \frac{1}{n}\sum_{t=1}^{n}\{\frac{1}{1+b^Tg_t(\lambda)}(\frac{\partial g_t(\lambda)}{\partial\lambda^T})^T - \frac{g_t(\lambda)b^T\frac{\partial g_t(\lambda)}{\partial\lambda^T}}{\{1+b^Tg_t(\lambda)\}^2}\} = \{\frac{\partial Q_{2n}(\lambda,b)}{\partial b^T}\}^T \xrightarrow{p} \Omega$$

定义

$$S_n = S_n(\lambda,b) = \begin{pmatrix} \frac{\partial Q_{1n}(\lambda,b)}{\partial b^T} & \frac{\partial Q_{1n}(\lambda,b)}{\partial\lambda} \\ \frac{\partial Q_{2n}(\lambda,b)}{\partial b^T} & \frac{\partial Q_{2n}(\lambda,b)}{\partial\lambda} \end{pmatrix}$$

且

$$S_n \xrightarrow{p} \begin{pmatrix} -\hat\Omega & \Omega \\ \Omega^T & 0 \end{pmatrix} \triangleq \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$$

应用泰勒展式，式中 $S_{22.1}$ 定义形式同定理 1[11-12]。

且

$$Q_{1n}(\hat\lambda_n,b(\hat\lambda_n)) = 0,\ Q_{2n}(\hat\lambda_n,b(\hat\lambda_n)) = 0$$

得到

$$\begin{pmatrix} b(\hat\lambda_n) \\ \hat\lambda_n - \lambda_0 \end{pmatrix} = S_n^{-1}(\lambda^* \quad b^*)\begin{pmatrix} -Q_{1n}(\lambda_0,0) \\ 0 \end{pmatrix} + o_p(1)$$

其中，λ^* 是 $\hat\lambda_n$ 与 λ_0 范围内相应的值，b^* 是 $b(\hat\lambda_n)$ 与 0 范围内相应值。

$$\ln(r;\theta_0,\hat\lambda_n) = -nQ_{1n}^T(\lambda_0,0)S_{11}^{-1} \times \{1 + S_{12}S_{22.1}^{-1}S_{21}S_{11}^{-1}\}Q_{1n}(\lambda_0,0) + o_p(1) =$$

$$\{(-S_{11})^{\frac{-1}{2}}\sqrt{n}Q_{1n}(\lambda_0,0)\}^T\{1 - (-S_{11})^{\frac{-1}{2}}S_{12}S_{22.1}^{-1}S_{21}(-S_{11})^{\frac{-1}{2}}\} \times \{(-S_{11})^{\frac{-1}{2}}\sqrt{n}Q_{1n}(\lambda_0,0)\} + o_p(1)$$

易知

$$(-S_{11})^{\frac{-1}{2}}\sqrt{n}Q_{1n}(\lambda_0,0)$$

服从多维标准正态分布

$$1 - (-S_{11})^{\frac{-1}{2}}S_{12}S_{22.1}^{-1}S_{21}(-S_{11})^{\frac{-1}{2}}$$

对称且秩为 1。可见

$$l_n(r;\theta_0;\hat\lambda_n(\theta_0)) \xrightarrow{d} \chi^2(1)$$

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Empirical Likelihood Estimation of GARCH Model

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Abstract: The *GARCH* models based on *ARCH* models in economics are discussed. The main work of this article is as follows: the empirical likelihood estimation of β-*ARCH* model and *GARCH* model are briefly discussed. Then, the large sample properties have been validated.

Key words: auto-regressive conditional hetero scedasticity model (*ARCH*); general auto-regressive conditional hetero scedasticity model (*GARCH*); empirical likelihood estimation