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On Kirichenko Tensors of Nearly-Kählerian Manifolds

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Abstract: A short description of structural and virtual Kirichenko tensors that form a complete system of first-order differential-geometrical invariants of an arbitrary almost Hermitian structure is given. A characterization of nearly-Kählerian structures in terms of Kirichenko tensors is also given. Keywords

Key words: Kirichenko tensors; Ricci tensor; nearly-Kählerian manifold; almost Hermitian manifold; six-dimensional submanifolds of Cayley algebra

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I INTRODUCTION

The almost Hermitian structures (AH-structures) belong to the most substantial differential-geometrical structures. A great number of signification publications have been devoted to the study of such structures. These works characterize almost Hermitian structures from the point of view of differential geometry as well as of modern theoretical physics. Such well-known mathematicians as R. Brown, E. Calabi, N. Ejiri, A. Gray, L. M. Hervella, K.-T. Kim, T. Koda, M. Prvanovic, G. Rizza, K. Sekigawa, I. Vaisman, L. Vanhecke, K. Yano made a great contribution to the theory of almost Hermitian structures. No doubt that one of the first places in this list of names will be occupied by the Russian geometer Vadim Feodorovich Kirichenko who has obtained a set of important results in this field [1-5]. He was one of the fist scientists to use systematically the method of associated G-structures. This method is modern variant of the Cartan's exterior form method [6] developed by G. F. Laptev and A. M. Vasil'ev. Before V. F. Kirichenko, most significant investigation on almost Hermitian structures was done in terms of Koszul's invariant calculation[7]. Without denying the effective Koszul's calculation system, V. F. Kirichenko and his pupils (A. Abu-Saleem, I. Borisovsky, A. Gritsans, A. Rustanov, A. Shihab, L. Stepanova, E. Volkova, B. Zayatuev and others) have obtained the principal part of their results precisely by the method of associated G-structures.

In [2], V. F. Kirichenko has obtained the first group of Cartan structural equations of an almost Hermitian manifold on the space of the associated G-structure. V. F. Kirichenko has introduced the notions of structural and virtual tensors that form a complete system of first-order differential-geometrical invariants of an almost Hermitian structure. In the present paper, we give a short description of these tensors and characterize the class of nearly-Kählerian manifolds in terms of Kirichenko tensors.

II PRELIMINARIES

We consider an almost Hermitian manifold, i. e. a 2n dimensional manifold M^{2n} with a Riemannian metric $g=\langle\cdot,\cdot\rangle$ and an almost complex structure J. Moreover, the following condition must hold

$$\langle JX, JY\rangle = \langle X, Y\rangle, X, Y \in \aleph(M^{2n})$$

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where $\aleph(M^{2n})$ is the module of smooth vector fields on M^{2n}. All considered manifolds, tensor fields and similar objects are assumed to be of the class C^{∞}.

The specification of an almost Hermitian structure on a manifold is equivalent to the setting of a G-structure, where G is the unitary group $U(n)$ [2-3]. Its elements are the frames adapted to the structure (A-frames). They look as follows:

$$(p,\varepsilon_1,\cdots,\varepsilon_n,\varepsilon_{\hat{1}},\cdots,\varepsilon_{\hat{n}})$$

where ε_a are the eigenvectors corresponded to the eigenvalue $i=\sqrt{-1}$, and $\varepsilon_{\hat{a}}$ are the eigenvectors corresponded to the eigenvalue $-i$. Here and further the index a ranges from 1 to n, and we state $\hat{a}=a+n$.

Therefore, the matrixes of the operator of the almost complex structure and of the Riemannian metric written in an A-frame look as follows, respectively:

$$(J_j^k)=\left(\begin{array}{c|c} iI_n & 0 \\ \hline 0 & -iI_n \end{array}\right) \qquad (g_{kj})=\left(\begin{array}{c|c} 0 & I_n \\ \hline I_n & 0 \end{array}\right) \qquad (1)$$

where I_n is the identity matrix; k, j = 1,⋯, 2n.

We recall that the fundamental form (or Kählerian form) of an almost Hermitian manifold is determined by the relation

$$F(X,Y)=\langle X,JY\rangle, X,Y\in \aleph(M^{2n})$$

By direct computing it is easy to obtain that in A-frame the fundamental form matrix looks as follows:

$$(F_{kj})=\left(\begin{array}{c|c} 0 & iI_n \\ \hline -iI_n & 0 \end{array}\right)$$

It is expedient to consider the other tensors written in an A-frame. This corresponds to the problems of the study of almost Hermitian manifolds. We remark that the Russian mathematician V. F. Kirichenko worked out such a method[2].

III KIRICHENKO TENSORS

The form of Levi-Civita connection ∇ is determined by the forms system $\{\omega_j^k\}$ on the space of the frames stratification over an almost Hermitian manifold M^{2n}. Similarly, the displacement form ω is determined by the forms system $\{\omega^j\}$. The Cartan structural equations on the stratification space over almost Hermitian manifold look as follows [3]:

$$d\omega^k=\omega_j^k\wedge\omega^j$$
$$d\omega_j^k=\omega_m^k\wedge\omega_j^m+\frac{1}{2}R^k\omega^m\wedge\omega^l \qquad (2)$$

where $\{R_{jml}^k\}$ are components of the Riemannian curvature tensor (or of the Riemann-Christoffel tensor[6]). Here and further $k,j,l,m=1,...,2n$.

As the complex structure J and the Riemannian metric g are tensors of the (1,1) and (2,0) type, respectively, and as $\nabla g=0$, then the components of these tensors must satisfy the following system of differential equations:

$$dJ_j^k+J_l^k\omega_j^l-J_j^l\omega_l^k=J_{j,l}^k\omega^l$$
$$dg_{kl}+g_{lj}\omega_k^l+g_{kl}\omega_j^l=0 \qquad (3)$$

where $\{J_{l,l}^k\}$ are the components of ∇J. Tacking into account Eq. (1), we can rewrite the first of Eq. (3) as follows:

$$\omega_{\hat{b}}^a=-\frac{i}{2}J_{\hat{b},k}^a\omega^k;\omega_b^{\hat{a}}=-\frac{i}{2}J_{b,k}^{\hat{a}}\omega^k$$
$$\omega_b^{\hat{a}}=-\frac{i}{2}J_{b,k}^{\hat{a}}\omega^k;J_{b,k}^a=0;J_{\hat{b},k}^{\hat{a}}=0 \qquad (4)$$

Similarly, from the second of Eq. (3) we obtain:

$$\omega_{\hat{b}}^a+\omega_{\hat{a}}^b=0;\omega_b^a+\omega_{\hat{a}}^{\hat{b}}=0;\omega_b^{\hat{a}}+\omega_a^{\hat{b}}=0 \qquad (5)$$

Here and further $a,b,c,d=1,...,n;\hat{a}=a+n$.

Substituting Eq. (4) and Eq. (5) in Cartan structural Eq. (2), we have:

$$d\omega^a=\omega_b^a\wedge\omega^b+\omega_{\hat{b}}^a\wedge\omega^{\hat{b}}=$$
$$\omega_b^a\wedge\omega^b-\frac{i}{2}J_{\hat{b},c}^a\omega^c\wedge\omega^{\hat{b}}-\frac{i}{2}J_{[\hat{b},\hat{c}]}^a\omega^{\hat{c}}\wedge\omega^{\hat{b}}$$
$$d\omega^{\hat{a}}=\omega_b^{\hat{a}}\wedge\omega^b+\omega_{\hat{b}}^{\hat{a}}\wedge\omega^{\hat{b}}= \qquad (6)$$
$$\omega_{\hat{b}}^{\hat{a}}\wedge\omega^{\hat{b}}-\frac{i}{2}J_{b,\hat{c}}^{\hat{a}}\omega^{\hat{c}}\wedge\omega^b-\frac{i}{2}J_{[b,c]}^{\hat{a}}\omega^c\wedge\omega^b$$

We denote

$$\omega_k=g_{kj}\omega^j$$

In particular,

$$\omega_a=\omega^{\hat{a}}=\overline{\omega^a}$$

Taking into account this fact as well as Eq. (5), we can rewrite Eq. (6) as follows:

$$d\omega^a=\omega_b^a\wedge\omega^b+B^{ab}\omega^c\wedge\omega_b+B^{abc}\omega_b\wedge\omega_c$$
$$d\omega_a=-\omega_a^b\wedge\omega_b+B_{ab}\omega_c\wedge\omega^b+B_{abc}\omega^b\wedge\omega^c \qquad (7)$$

where

$$B^{ab}{}_c=-\frac{i}{2}J_{\hat{b},c}^a; \qquad B_{ab}{}^c=\frac{i}{2}J_{b,\hat{c}}^{\hat{a}}$$
$$B^{abc}=\frac{i}{2}J_{[\hat{b},\hat{c}]}^a; \qquad B_{abc}=-\frac{i}{2}J_{[b,c]}^{\hat{a}} \qquad (8)$$

Considering the differential continuations of Eq. (8) it is not difficult to see that

$$dB_c^{ab}+B_d^{ab}\omega_c^d-B_c^{db}\omega_d^a-B_c^{ad}\omega_d^b=B_{c,k}^{ab}\omega^k$$
$$dB_{ab}{}^c-B_{ab}{}^d\omega_d^c+B_{db}{}^c\omega_a^d+B_{ad}{}^c\omega_b^d=B_{ab}{}^c{}_{,k}\omega^k$$
$$dB^{abc}-B^{dbc}\omega_d^a-B^{adc}\omega_d^b-B^{abd}\omega_d^c=B_{,k}^{abc}\omega^k$$
$$dB_{abc}+B_{dbc}\omega_a^d+B_{adc}\omega_b^d+B_{abd}\omega_c^d=B_{abc,k}\omega^k$$

That is why we conclude that the system of functions

$\{B^{ab}_{c}\}$, $\{B_{ab}{}^{c}\}$, $\{B^{abc}\}$, $\{B_{abc}\}$

forms a set of the components of a complex tensors on an almost Hermitian manifold M^{2n}.

Definition 1[8]. The tensors with components $\{B^{ab}_{c}\}$ and $\{B_{ab}{}^{c}\}$ are called virtual Kirichenko tensors of first and second order, respectively (or KV-tensors).

Definition 2[8]. The tensors with components $\{B^{abc}\}$ and $\{B_{abc}\}$ are called structural Kirichenko tensors of first and second order, respectively (or KS-tensors).

We remark that according to Eq. (6)

$$J^{a}_{\hat{b},c} + J^{b}_{\hat{a},c} = 0$$

So, we have

$$B^{ab}_{c} + B^{ba}_{c} = 0$$

Similarly

$$B_{ab}{}^{c} + B_{ba}{}^{c} = 0$$

Thus, we have proved.

Proposition 1. The Kirichenko virtual tensors of an almost Hermitian manifold are skew-symmetric on the first pair if indices.

From Eq. (8) we obtain the following result:

Proposition 2. The Kirichenko structural tensors of an almost Hermitian manifold are skew-symmetric on the second pair if indices.

Owing to the reality of ∇J, from the given definition we have:

Proposition 3. The Kirichenko tensors of first and second order of an almost Hermitian manifold are conjugate, i. e.

$$\overline{B^{ab}{}_{c}} = B_{ab}{}^{c}, \quad \overline{B^{abc}} = B_{abc}$$

IV MAIN RESULTS

As it is known[3,9], an almost Hermitian structure is nearly-Kählerian if and only if the following condition is fulfilled:

$$\nabla_{X}(F)(X,Y) = 0$$

Using the definitions of Kirichenko tensors, by direct computing we can reformulate this condition in terms of Kirichenko tensors.

THEOREM 1. An almost Hermitian structure is nearly-Kählerian if and only if

$$B^{abc} = -B^{bac}, B_{abc} = -B_{bac}$$

$$B^{ab}_{c} = 0, \quad B_{ab}{}^{c} = 0$$

That is why we can rewrite the Cartan structural equations for the case when the almost Hermitian structure is nearly-Kählerian.

THEOREM 2. The first group of Cartan structural equations of a nearly-Kählerian structure is the following:

$$d\omega^{a} = \omega^{a}_{b} \wedge \omega^{b} + B^{abc}\omega_{b} \wedge \omega_{c}$$

$$d\omega_{a} = -\omega^{b}_{a} \wedge \omega_{b} + B_{abc}\omega^{b} \wedge \omega^{c}$$

where $B^{abc} = -B^{bac}, B_{abc} = -B_{bac}$.

We remark that the Kirichenko tensors were used mainly for characterization of Hermitian manifolds, i. e. of almost Hermitian manifolds with an integrable almost Hermitian structure (for instance, [10-13]). Namely, the THEOREMS 1 and 2 have analogs for Hermitian structures:

Proposition 4[8]. An almost Hermitian structure is Hermitian (i. e. an integrable almost Hermitian) if and only if its structural Kirichenko tensors vanish:

$$B^{abc} = 0, B_{abc} = 0$$

Proposition 5 [8]. The first group of Cartan structural equations of a Hermitian structure is the following:

$$d\omega^{a} = \omega^{a}_{b} \wedge \omega^{b} + B^{ab}_{c}\omega^{c} \wedge \omega_{b}$$

$$d\omega_{a} = -\omega^{b}_{a} \wedge \omega_{b} + B_{ab}{}^{c}\omega_{c} \wedge \omega^{b}$$

The most important example of nearly-Kählerian manifolds are the six-dimensional submanifolds of Cayley algebra. Such six-dimensional nearly-Kählerian manifolds were studied by E. Calabi [14], A. Gray[15-20], V. F. Kirichenko [21-22], Haizhong Li and Guoxin Wei[23-24], H. Hashimoto[25-27], K. Sekigawa[28], L. Vranchen[29], N. Ejiri[30], S. Funabashi and J. S. Pak[31] and others.

Knowing the expessions of Kirichenko tensors for such six-dimensional submanifolds of Cayley algebra[8], we obtain the following result.

THEOREM 3. The first group of Cartan structural equations of a six-dimensional nearly-Kählerian submanifold of Cayley algebra is the following:

$$d\omega^{a} = \omega^{a}_{b} \wedge \omega^{b} + \mu\varepsilon^{acb}\omega_{b} \wedge \omega_{c}$$

$$d\omega_{a} = -\omega^{b}_{a} \wedge \omega_{b} + \bar{\mu}\varepsilon_{acb}\omega^{b} \wedge \omega^{c}$$

where $\varepsilon_{abc} = \varepsilon^{123}_{abc}$ and $\varepsilon^{abc} = \varepsilon^{abc}_{123}$ are the components of the third-order Kronecher tensor.

These structural equations contain all information about the geometry of such six-dimensional submanifolds of Cayley algebra. We can apply this result to detailed study of such six-dimensional submanifolds of octonions algebra. For instance, taking into account the THEOREM 3, it is possible to use the recent Kirichenko's and Shihab's properties of conharmonic curvature tensor[32] in the theory of six-dimensional nearly-Kählerian submanifolds of Cayley algebra. Namely, Kirichenko and Shihab have proved that every conharmonically parakählerian nearly-Kählerian manifold is manifold of non-negative constant curvature. Moreover, such a manifold is a manifold of zero scalar curvature precisely when it is Kählerian[32]. Taking into account the classical Kirichenko properties of nearly-Kählerian structures[21], from these statements we conclude that a Kählerian manifold of dimension grater than 4 is conharmonically parakählerian if and only if it is Ricci-flat. Using also the classification of six-dimensional Kählerian submanifolds of Cayley algebra[22], we deduce the following results:

THEOREM 4. A nearly-Kählerian six-dimensional submanifold of Cayley algebra is a conharmonically parakählerian if and only if it is a Ricci-flat Kählerian manifold.

THEOREM 5. A nearly-Kählerian six-dimensional submanifold of Cayley algebra is conharmonically flat if and only if it is holomorphically isometric to the complex Euclidean space C^3 with a canonical Kählerian structure.

THEOREM 6. A simply connected nearly-Kählerian six-dimensional submanifold of Cayley algebra is a manifold of pointwise constant holomorphic conharmonic curvature if and only if it is a manifold of global constant holomorphic conharmonic curvature.

We remark that from THEOREM 5 it follows that the well-known Kirichenko's examples[21] of nearly-Kählerian six-dimensional submanifolds of Cayley algebra are not conharmonically flat. We obtain the following:

Proposition 6. Every six-dimensional Kählerian minimal surface in the Cayley algebra is not a conharmonically flat manifold.

The class of nearly-Kählerian manifolds contains the class of Kählerian manifolds. On the other hand, this class is a subclass of the class of $W_1 \oplus W_4$ -manifolds (using A. Gray and L. M. Hervella's notation[9]), or the class of Vaisman-Gray manifolds[3]. We remark that the manifolds of this class are the generalization of the class of nearly-Kählerian manifolds as well as of the class of locally conformal Kählerian manifolds. That is why at the end of this paper we pose the following open problem.

PROBLEM. Find a characterization in terms of Kirichenko tensors for Vaisman-Gray manifolds.

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